An Optimal Bound for the MST Algorithm to Compute Energy Efficient Broadcast Trees in Wireless Networks

Christoph Ambühl
Istituto Dalle Molle di Studi sull' Intelligenza Artificiale
Galleria 2, 6928 Manno, SWITZERLAND
christoph@idsia.ch

Technical Report No. IDSIA-22-04
November 2004

IDSIA / USI-SUPSI
Istituto Dalle Molle di studi sull'intelligenza artificiale
Galleria 2, 6928 Manno, Switzerland

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Abstract

Computing energy efficient broadcast trees is one of the most prominent operations in wireless networks. For stations embedded in the Euclidean plane, the best analytic result known to date is a 7.2-approximation algorithm based on computing an Euclidean minimum spanning tree. We improve the analysis of this algorithm and show that its approximation ratio is 6, which matches the previously known lower bound for this algorithm.

1 Introduction and Previous Work

Multi-hop wireless networks [10] require neither fixed, wired infrastructure nor predetermined interconnectivity. In particular, ad hoc networks [8, 12] are the most popular type of multi-hop wireless networks. An ad hoc wireless network is built of a bunch of radio stations. The links between them are established in a wireless fashion using the radio transmitters and receivers of the stations.

In order to send a message from a station s to a station t, station s needs to emit the message with enough power such that t can receive it. In the model, the power P_s required by a station s to transmit data to station t must satisfy the inequality

$$\frac{P_s}{\text{dist}(s,t)^\alpha} \; > \; \gamma. \tag{1}$$

The term dist(s,t) denotes the distance between s and t, and $\alpha \geq 1$ is the **distance-power gradient**, and $\gamma \geq 1$ is the **transmission-quality** parameter. In an ideal environment (i.e. in the empty space) it

*Supported by the Swiss National Science Foundation project 200021-100539/1, "Approximation Algorithms for Machine Scheduling Through Theory and Experiments". This paper contains ideas obtained while the author stayed at Università degli Studi di Roma " Tor Vergata" in Italy and was supported by the European Union under the RTN Project ARACNE.

holds that $\alpha = 2$ but it may vary from 1 to more than 6 depending on the environment conditions of the location of the network (see [13]).

In ad hoc networks, a power value is assigned to each station. These values, according to Equation (1), determine the so-called **range** of each station. The range of a station s is the area in which stations can receive all messages sent by s.

Using the ranges, one can determine the so-called **transmission graph** $G = (S, A)$. The vertex set S is the set of stations, and the directed edge from s to t is in A if and only if t is within the range of s.

All stations in the range of a station i can receive messages sent by i. The minimal range needed for station i to establish all its out-going connections in G is therefore

$$r_G(i) := \max_{j \in \Gamma_G(i)} \text{dist}(i, j). \tag{2}$$

where $\Gamma_G(i)$ denotes the set of out-neighbors of station i in G. The total power needed to establish all connections in G is therefore

$$\text{power}(G) := \sum_{i \in V} \gamma \cdot r_G(i)^\alpha, \tag{3}$$

Since the value of γ does not influence the relative quality of the solutions, we assume $\gamma = 1$ for the rest of the paper. In this paper we address the following problem:

Problem 1 (Energy Efficient Broadcast Tree Problem (EEBT)) *Let S be a set of stations represented by points from the Euclidean plane. That is, the distance function becomes dist$(s, t) := \|st\|$, where $\|st\|$ is the Eulidean distance between s and t. One of the stations is called the source station s. The goal is to find the transmission graph G which minimizes power(G) and contains a directed spanning tree rooted at s (a branching from s).*

The relevance of this problem is due to the fact that any transmission graph satisfying the above property allows the source station to perform a **broadcast** operation. Broadcast is a task initiated by the source station to transmit a message to all stations in the wireless network: This task constitutes a major part of real life multi-hop radio networks [1, 2, 5].

The **EEBT** Problem is known to be $\mathcal{NP}$-hard [3]. If the points are from Euclidean space, it is even known to be $\mathcal{APX}$-hard [3]. Furthermore, if the dist function is arbitrary, the problem cannot be approximated with a logarithmic factor unless $\mathcal{P} = \mathcal{NP}$ [7]. The currently best approximation algorithm for the **EEBT** Problem is as follows.

Algorithm 2 (ALG) *The input of the algorithm is a set of stations S represented by points in the Euclidean plane. One of the stations is designated as the source. The algorithm first computes the Euclidean minimum spanning tree (EMST) of the point set S. Then the EMST is turned into a directed EMST by directing all the edges such that there exists a directed path from the source station to all other stations.*

Recently, Flammini,Klasing, Navarra, and Perennes [6] showed that $\mathbf{ALG}$ is a 7.6-approximation algorithm. In [14], Wan, Călinescu, Li, and Frieder showed that $\mathbf{ALG}$ is a 12-approximation. Independently, Clementi, Crescenzi, Penna, Rossi, and Vocca showed an approximation ratio of 20 for $\mathbf{ALG}$ [3]. In this paper, we show that $\mathbf{ALG}$ is a 6-approximation for all $\alpha \geq 2$. This matches the lower bound given in [3] and [14].

In [11], exact algorithms for **EEBT** have been studied. The **EEBT** problem falls into the class of so-called **range assignment problems**: Find a transmission range assignment such that the corresponding transmission graph G satisfies a given connectivity property Π, and power(G) is minimized (see for example [9, 5]). In [4], the reader may find an exhaustive survey on previous results related to range assignment problems.

2 A Reduction to Upper Bounding EMSTs in the Unit Disc

Theorem 3 *Let S be a set of points from the unit disc, with $(0,0) \in S$. Let $e_1, e_2, \ldots, e_{|S|-1}$ be the edges of the Euclidean minimum spanning tree of S. Then*

$$\mu(S) := \sum_{i=1}^{|S|-1} \|e_i\|^2 \leq 6.$$

Lemma 4 *A bound on $\mu(S)$ automatically implies the same bound on the approximation ratio of* $\mathbf{ALG}$.

Theorem 3 is the main theorem if this paper. The main purpose of this section is to prove Lemma 4. Up to a few differences concerning the station at the origin of the unit disc, the proof was already given in [14] to obtain the 12-approximation.

Remember that for power(G), only the largest out-going edge of each station is considered. An simple upper bound on power(G) is therefore

$$\text{tot}(G) := \sum_{i \in S} \sum_{j \in \Gamma_G(i)} \|ij\|^\alpha.$$

Hence, instead of adding up only the most expensive out-going edge of each vertex as in power(G), we sum up the cost of all the edges of the MST. Note that tot(G) does not depend on the directions of the edges anymore, since we just sum up all of them.

Let us define mst(S) := tot(G) where G is the MST returned by $\mathbf{ALG}$ on input S. Let opt(S) be the cost of the optimal range assignment for a set of stations S. In order to determine the worst case approximation quality of $\mathbf{ALG}$ we need to determine an upper bound on

$$\max_S \frac{\text{mst}(S)}{\text{opt}(S)}. \tag{4}$$

Lemma 5 *Among the instances S that maximize Equation (4), there is one for which in the optimal solution, the source station covers all the other stations. That is, all stations except the source have range zero in the optimal solution.*

Proof. Let S be any instance that maximizes Equation (4) and let $\hat{G}$ be an optimal communication graph for S. Remember that $\Gamma_{\hat{G}}(v)$ is the set of out-neighbors of v in $\hat{G}$. Clearly we can bound

$$\text{mst}(S) \leq \sum_{v \in S} \text{mst}(\Gamma_{\hat{G}}(v)),$$

since all the edges of the later form a spanning graph, and the MST of S is the cheapest among them. It also holds

$$\frac{\text{mst}(S)}{\text{opt}(S)} = \frac{\text{mst}(S)}{\text{power}(\hat{G})} \leq \frac{\sum_{v \in S} \text{mst}(\Gamma_{\hat{G}}(v))}{\sum_{v \in S} r(v)^\alpha} \leq \max_{v \in S} \frac{\text{mst}(\Gamma_{\hat{G}}(v))}{r(v)^\alpha}.$$

Let $\hat{v}$ be a station that maximizes the fraction on the right. Clearly, $\Gamma_{\hat{G}}(\hat{v})$ is also a worst case example for $\mathbf{ALG}$.

Furthermore, it holds that the optimal solution for $\Gamma_{\hat{G}}(\hat{v})$ is to send the message directly from $\hat{v}$ to all stations in $\Gamma_{\hat{G}}(\hat{v})$ without intermediate hops. Otherwise, we would get a contradiction to the optimality of $\hat{G}$. □

Proof of Lemma 4. Let $\alpha = 2$. Consider a worst case instance $\Gamma_{\hat{G}}(\hat{v})$ as described in the proof of Lemma 5. Let T be the instance obtained by translating $\Gamma_{\hat{G}}(\hat{v})$ such that $\hat{v}$ becomes the origin and then applying a central delation [15] with center $\hat{v}$ such that the station furthest from $\hat{v}$ is at distance 1 from $\hat{v}$. Since $\text{opt}(T) = 1$, $\mu(T) = \text{mst}(T)$ and the ratio between mst and opt is unaffected by these operations, it holds

$$\mu(T) = \frac{\text{mst}(T)}{\text{opt}(T)} = \frac{\text{mst}(\Gamma_{\hat{G}}(\hat{v}))}{\text{opt}(\Gamma_{\hat{G}}(\hat{v}))}.$$

Hence for $\alpha = 2$, upper bounds on $\mu(T)$ indeed translate to upper bounds for $\mathbf{ALG}$. For $\alpha \geq 2$, we even have $\mu(T) \geq \text{mst}(T)/\text{opt}(T)$. This holds since $\text{opt}(T) = 1$ for all α and $\text{mst}(T)$ decreases with growing α, since all MST edges of T have length at most 1. □

We now sketch the proof of the $\mu(S) \leq 12$ bound given in [14]. It works as follows. The cost if each edge e of the MST is represented by a geometric shape called diamonds, show in Figure 1 on the bottom left. They consist of two isosceles triangles with an angle of $\frac{2}{3}\pi$. The area of a diamond for an edge e with length $\|e\|$ is $\lambda \cdot \|e_i\|^2$, with $\lambda = \sqrt{3}/6$. Diamonds are considered being open sets. It can be shown that if one puts these diamonds along the edges of an MST as shown in the middle of Figure 1, they do not intersect. It can further be shown that the area shown on the right of Figure 1 is the largest area that can be covered by the diamonds. Its area is 12λ. Therefore one can conclude $\mu(S) \leq 12\lambda/\lambda = 12$.

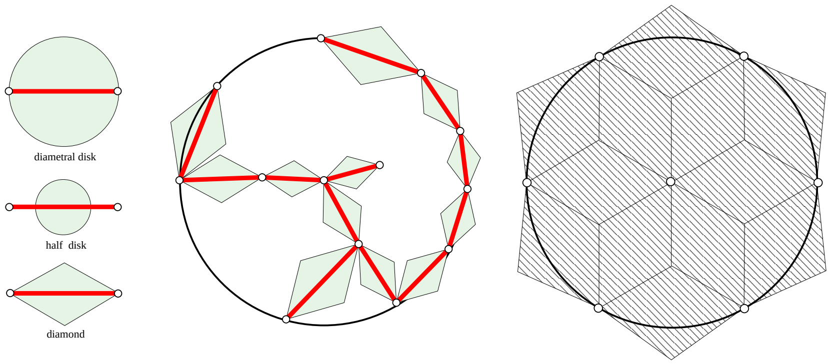

Figure 1: Proof idea of previous bounds.

Similar bounds can be obtained using diametral discs ($\mu(S) \leq 40$) or half discs ($\mu(S) \leq 20$) [3]. In both cases, one has to give an upper bound on the area generated by these shapes. In the case of diametral discs, this is done using the fact in any point, at most five diametral discs can intersect. This gives only a very crude bound, which leads too a very crude bound on $\mu(S)$. On the other hand, open half discs do not intersect. But since they are smaller than diamonds, the bound provided by them is worse.

Concerning lower bounds on $\mu(S)$, there is a point set S that attains $\mu(S) = 6$. It is a regular 6-gone with one point in the middle. A lower bound on the approximation ratio of $\mathbf{ALG}$ is shown in Figure 2. The length of the edges of the MST shown in Figure 2 are ε and $1-\varepsilon$, respectively. We have opt$(S) = 1$ and power$(G) = \varepsilon^2 + 6 \cdot (1-\varepsilon)^2$. Hence for $\varepsilon \to \infty$, the ratio between the two becomes 6. This lower bound holds for all values of α. Our analysis will give a matching upper bound for this lower bound for $\alpha \geq 2$. In the appendix it is shown that for $\alpha < 2$, $\mathbf{ALG}$ does not have a constant approximation ratio.

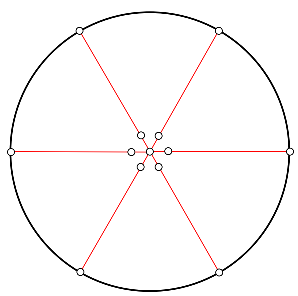

Figure 2: A worst case example for $\mathbf{ALG}$.

3 The Main Idea of the Proof of Theorem 3

Among the shapes that do not intersect, diamonds seem to be the best possible geometric shape for this kind of analysis. For a better bound, we need to use larger shapes and we need to deal with the intersection of the shapes more accurately than before. The shapes used for the new bound are pairs of equilateral triangles, one on each side of the MST edges. As depicted in Figure 3, the equilateral triangles intersect heavily. We will give a quite accurate bound on the area generated by them.

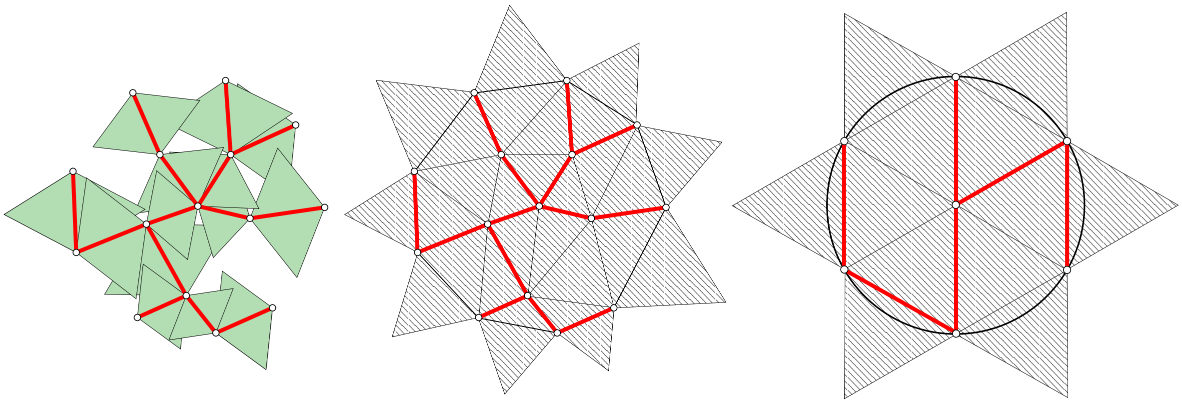

Figure 3: The total area of the equilateral triangles on the left is bounded by the hatched area in the middle. The point set that maximizes the hatched area is the star shown on the right.

A high level description of the proof of our bound is the following. Consider a point set S with n points. Hence, the MST will have $n-1$ edges and therefore, there will be $2(n-1)$ equilateral triangles representing the cost of the MST. Let A_{MST} be the total area generated by these triangles.

Let c be the number of edges of the convex hull of S. By triangulating S, we end up with a planar graph G with n vertices, e edges, and f facets. Let t be the number of triangles of the triangulation. Then the following three equations hold.

$$\left| \begin{array}{ccc} f & = & t+1 \\ 3t+c & = & 2e \\ n+f-e & = & 2 \end{array} \right|$$

The first one simply states that the number of facets is equal to the number of triangles plus the infinite facet. For the second one, we add up the number of edges of all facets. For triangles, this is 3, whereas for the infinite facet, it is c. Since every edge is part of exactly two facets, this sums up to $2e$. The last equation is the Descartes-Euler polyhedral formula [15]. If we solve the system for t, we obtain $t = 2(n-1) - c$. Hence, if we add c equilateral triangles along the convex hull of S as depicted in the center of Figure 3, the number of triangles becomes equal to the number of triangles generated by the MST, as shown on the left side of the figure. Let A_{TRI} be the total area of the triangles within the convex hull of S plus the c additional triangles along the convex hull.

The main idea of the proof is to show that $A_{\mathrm{MST}} \leq A_{\mathrm{TRI}}$. To get an intuitive understanding of it, consider a point set S obtained from the hexagonal grid for which all edges of the triangulation of its convex hull have the same length. In this case, all triangles that are involved in A_{MST} and A_{TRI} are congruent. Since their number is equal, it holds $A_{\mathrm{MST}} = A_{\mathrm{TRI}}$. Intuitively, if the edges of the triangulation have different lengths, A_{MST} will be smaller compared to A_{TRI} since the MST will be composed mainly of small edges.

We then conclude the proof by showing that A_{TRI} is maximized by the star configuration depicted on the left of Figure 3. The area of the star is 6λ. Therefore we get $\mu(S) \leq 6\lambda/\lambda = 6$.

4 A Sketch of the Proof of Theorem 3

First, we introduce some notations. The area of an equilateral triangle with side length s will be denoted by $\triangle(s)$. The area of a triangle with edge lengths a, b, and c is denoted by $\triangle(a, b, c)$. Every edge can be partitioned lengthwise into two **half edges**. Both half edges are incident to the same vertices, but each of them is incident to only one facet. Slightly abusing notations, we call the largest side of an obtuse triangle its **hypothenuse**.

Consider the two triangles incident to an edge e. Let α and β be the two angles opposite e in the two triangles. We will call β the **opposite angle of** α.

Consider the MST of S and the Delaunay triangulation of the convex hull of S. Remember that the MST edges are also edges of the Delaunay triangulation. Now choose any edge e of the triangulation. Consider the unique cycle that is formed by adding e to the MST. This cycle and its (finite) interior is called a **pocket**. The triangles of the Delaunay triangulation within a pocket are called **pocket triangles**. The **area of a pocket** is the total area of all pocket triangles. The edge e is called the **door** of the pocket. All MST edges of the cycle are called **border edges**. Those in the interior are called **interior edges**. If e is an MST edge, the pocket will be called an **empty pocket**. Here, e is a border edge and the door at the same time. Empty pockets have area 0.

Note that the door of a pocket is incident to exactly one pocket triangle. If this triangle is obtuse and the door is its hypothenuse, the pocket is called an **obtuse pocket**, otherwise we call it an **acute pocket**.

The **MST triangles of a pocket** P is the following set of triangles. Every half edge which is part of the MST and incident to a pocket triangle of P generates an MST triangle. An MST triangle of a half edge of length l is an equilateral triangle with side length l.

Obviously, both half edges of an interior edge belong to the pocket. On the other hand, only one half edge of a border edge belongs to the pocket. The **MST-area of a pocket** is the sum of the areas of all the MST triangles. The MST-area of an empty pocket is $\triangle(e)$.

Figure 4 shows a pocket. The door of the pocket is the dashed line. Its border consists of all the edges of the MST connecting the two end points of the pocket. The largest of these edges is denoted by b_1. There are four inner edges. Note that the inner edges have two MST triangles attached, one for each half edge, whereas the border edges have only one. The area of the pocket consists of the interior of the pocket. Because b_1 is part of the MST whereas the door is not, b_1 is never longer then the door.

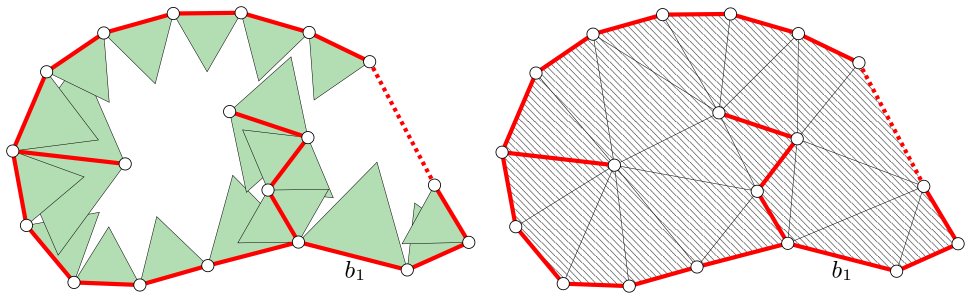

Figure 4: A pocket with its MST triangles on the left and the pocket triangles on the right. Note that there are 22 MST triangles and 21 pocket triangles.

Lemma 6 *In a acute pocket with largest border edge b, the difference between MST-area and pocket area is bounded by $\triangle(\|b\|)$.*

The proof of Lemma 6 is quite complicated. We therefore only give a sketch of it towards the end of this section. The full proof is given in the appendix. Using this lemma, one can prove Lemma 7. Due to lack to space, its proof is also given in the appendix. Lemma 7 in turn leads directly to the proof of Theorem 3.

Lemma 7 *Consider a pocket formed by an edge e. Then its MST-area can be bounded by the area of the pocket plus the area of a set of equilateral triangles whose side lengths are bounded by 1 and add up to $\|e\|$.*

Proof of Theorem 3. Consider the pockets whose doors are the edges of the convex hull of S. The sum of the MST areas of all these pockets is equal the total MST-area generated by S. Using Lemma 7, we can conclude that the total MST area is bounded by the area of a so-called sun. A sun is defined by a convex set T from the unit disc, with the additional property that all edges of the convex hull are bounded by 1. The **area of a sun** is the convex hull of T plus, for each edge e of the convex hull of T, the area of an equilateral triangle with side length $\|e\|$.

Observe that the MST area of S is $\mu(S) \cdot \sqrt{3}/2$. Hence we just need to prove that the area of a sun is bounded by $6\sqrt{3}/2$, which is exactly the area of a sun produced by a regular hexagon.

A point set T maximizing the area of its sun has all points on the unit circle. This holds since by moving a point towards the unit circle, the area of the sun increases. The area of a sun with all points on the unit circle can be partitioned as indicated in Figure 5(a). Each part consists of the triangle

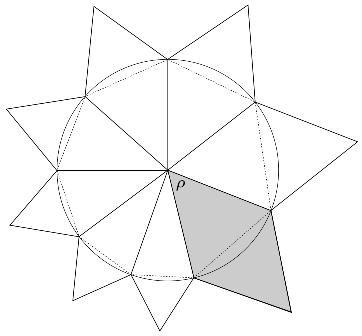

Figure 5(a): A sun.

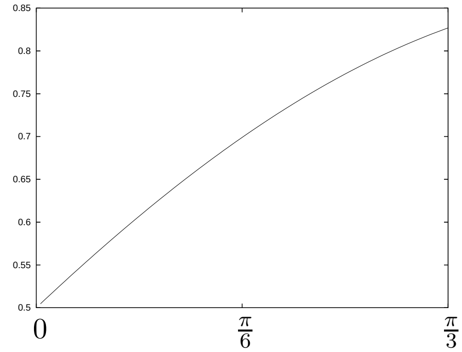

Figure 5(b): $f(\rho) = \frac{\frac{1}{2}sin(\rho)+\sqrt{3}sin(\frac{1}{2}\rho)^2}{\rho}$

formed by the origin and an edge of the convex hull, plus the corresponding equilateral triangle. The area of each part can be expressed in terms of the angle ρ the first triangle forms at the origin. It is

$$f(\rho) := \sin\left(\frac{\rho}{2}\right)\left(\cos\left(\frac{\rho}{2}\right) + 2\cdot\sin\left(\frac{\rho}{2}\right)\frac{\sqrt{3}}{2}\right) = \frac{1}{2}sin(\rho) + \sqrt{3}\cdot sin\left(\frac{\rho}{2}\right)^2.$$

Because we assumed that the edges of the convex hull are bounded by 1, the angle ρ must be between 0 and $\frac{\pi}{3}$. Note that $f(\frac{\pi}{3}) = \sqrt{3}/2$. In order to prove that the sun area is maximized by a regular hexagon, observe from Figure 5(b) that $f(\rho)/\rho$, restricted to the range $0 \le \rho \le \frac{\pi}{3}$, is maximized for $\rho = \frac{\pi}{3}$. □

In the remainder of this section, we describe the main ideas of the proof of Lemma 6.

Lemma 8 *In a pocket, the number of MST-triangles exceeds the number of pocket triangles by one.*

Proof. Let n, e, b, t, be the number of nodes, edges, border edges (not including the door), and pocket triangles of a pocket. Let us first count the MST triangles. Since the edges of a pocket form a tree, there are $n-1$ edges of the MST involved in the pocket. Each border edge produces one MST triangle, whereas each inner edges produces two. Therefore their number is $2n-2-b$. Let us now count the pocket triangles. The Descartes-Euler polyhedral formula gives $n+(t+1)-e=2$, where the additional 1 is the face outside the pocket. By double counting the half edges, we obtain $2e = 3t + (b+1)$. Here, the additional 1 stands for the door of the pocket. Solving for t, we get that the number of pocket triangles is $t = 2n-2-b-1$. □

The **extended pocket area (EP-area)** of a pocket is defined as the area of the pocket triangles plus an additional equilateral triangle with side length b, where b is the longest border edge of the pocket.

This additional triangle is called **door triangle**. We call the union of the pocket triangles and the door triangle **EP-triangles**. The **net-area** of the pocket is defined as its EP-area minus its MST-area. Using the above definition, Lemma 6 can be rewritten as

Lemma 6 (reformulated) *The net-area of an acute pocket is non-negative.*

Let $V \subseteq S$ be the set of vertices that are part of the pocket, i.e., all the vertices that are inside or at the border of the pocket. Let $G = (V, E)$ be the weighted graph obtained by adding all the edges of the triangulation of the pocket, including the border edges and the door. The weight of the edge $e = uv$, $u, v \in S$, is denoted by $w(e)$ and its value is $\|uv\|$. Observe that the EP-, MST-, and net-area of a pocket are just a sum of triangle areas. Therefore, using Heron's formula [15]

$$\triangle(a, b, c) = \frac{1}{4}\sqrt{-a^4 - b^4 - c^4 + 2\,a^2b^2 + 2\,a^2c^2 + 2\,b^2c^2}$$

for the area of a triangle with side lengths a, b, and c, they can all be expressed in terms of the weighted planar graph G. What is more, defining EP-, MST-, and net-area in terms of weighted planar graphs allows to define them even if the planar graph does not have an embedding in the plane. The next lemma takes advantage of this fact.

Lemma 9 *Let G' be a graph obtained from G by setting all weights to the same value. Then the net-area of G' is 0.*

Proof. If all weights of G' are equal, all the triangles involved in the EP-area and the MST-area of G' are equilateral and have the same side length. By Lemma 8, the number of EP-triangles is equal to the number of MST triangles. Since both the EP-area and the MST-area consist of the same number of congruent triangles, we can conclude that their area is equal. □

We will now define a continuous process that turns G into a graph in which all edges have the same weight. During the process, only the weights of the edges are altered, whereas the combinatorial structure of G remains unchanged. The process is designed in such a way that the net-area of G decreases monotonically. This property, together with Lemma 9, proves that the net-area of G is non-negative.

Let $w_{\min}$ and $w_{\max}$ be the length of the smallest and the largest edge in G. The process will be described by a set of graphs $G(m)$, $m \in \mathbb{R}$, $w_{\min} \leq m \leq w_{\max}$. We start with $G = G(w_{\max})$ and end with $G(w_{\min})$, in which all weights will be $w_{\min}$.

The complete proof is quite involved and therefore moved to the appendix. In the remained of the paper, we sketch the proof for a special case. Namely, we assume that all pocket triangles in G are acute. In this case, the process can be described very easily: Let $w(e)$ and $w_m(e)$ be the weight of edge e in G and $G(m)$ respectively. Then $w_m(e) = \min(m, w(e))$. That is, in every stage of the process, all maximal edges are decreased simultaneously until all edges have the same weight.

During this process only maximal edges are decreased. Hence, the ordering of the edges in terms of length remains unchanged during the process. Therefore the MST of G remains valid in all $G(m)$.

It is easy to see that during the process, the area of the pocket triangles of G decrease monotonically. This holds only because we assumed that the pocket triangles are acute. Handling obtuse triangles is very complicated and is deferred to the appendix.

We need to show that the net-area decreases. Hence, we have to show that in every $G(m)$, the decrease of the pocket area is at least as large as the decrease of the MST-area. To do this, we will partition $G(m)$ into so-called **chains** for which we will prove that their total net-area decreases monotonically.

Consider a graph $G(m)$ for fixed m. Chains are defined in terms of a graph Q. The vertex set of Q is the set of triangles in $G(m)$ plus the door triangle. Each maximal MST edge e of $G(m)$ creates the following set of edges in Q. If we remove e, the MST is divided into two subtrees. Let $R(e)$ be the ring of triangles that separates the two subtrees. For any pair of adjacent triangles in $R(e)$, we add an edge in Q. Note that if e is a border edge, the door triangle is also part of the ring and it is connected with the triangle incident to e and the pocket triangle incident to the door, as shown on the right of Figure 5. This completes the definition of the graph Q.

The **chains** are defined as the connected components of Q. Let Q' be a chain. We can define the area, the MST-area, and the net-area of a chain as follows. The **area of a chain** is equal to the sum of the areas of all triangles of Q'. Concerning the MST-area, note that every MST half edge h of $G(m)$ and its corresponding MST-triangle can be assigned to a unique chain, namely the one that contains the unique triangle h is incident to. The **MST area of a chain** is equal to the sum of the areas of all MST triangles belonging to the chain. The **net-area of a chain** is its area minus its MST-area. Since the net-area of a pocket is equal to the sum of the net-areas of all its chains, all we have to do to complete the proof is to show that the net-area of a chain decreases monotonically.

Remember that in $G(m)$, only the maximal MST edge decrease. Therefore only the MST-triangles of maximal MST edges decrease. In what follows, we will show that the decrease of the area of the chain makes up for the decrease of these **maximal MST-triangles**.

Some chains contain only a single triangle and no maximal MST-triangles. For these chains, it is obvious that the net-area decreases. Let us now look at chains that contain maximal MST-triangles. We need to consider two cases.

In case (i), we assume that the door triangle is not part of the chain. The **border of** Q' is the cycle in G with smallest area that contains all the triangles of Q'. The triangles from Q' that are incident to a border edge are called **border triangles**. On the left of Figure 5, the border triangles are shaded.

Let us now change Q' as follows. Let e be a maximal MST edge belonging to Q'. Assume its two incident triangles are q_1 and q_2. Add two new vertices h_1 and h_2 to Q'. They represent the half edges of e. Then remove the edge q_1q_2 from Q' and add q_1h_1 and q_2h_2. If we do this for all maximal MST edges e, Q' becomes a tree. Let d_i be the number of vertices of Q' with degree i, let n and e be the number of vertices and edges, respectively. From $e = n - 1$ (since Q' is a tree), $n = d_1 + d_2 + d_3$ and $2e = 3d_3 + 2d_2 + d_1$ (by double counting), we get $d_3 = d_1 - 2$.

The vertices of degree three represent equilateral triangles with side length m in $G(m)$. Let D_3

be the set of these triangles. The vertices of degree one represent the maximal MST half edges in $G(m)$. Let D_1 be the set of their corresponding MST-triangles. The area of the triangles in D_1 and D_3 decreases in the same way. Hence, the decrease of all but two triangles from D_1 is made up by the triangles in D_3. To complete the proof, one can show that the decrease of the border triangles of the chain makes up for the decrease of the remaining two maximal MST triangles. Due to lack of space, we skip this proof here.

For case (ii), assume that the door triangle belongs to Q'. Apply the edge splitting of the previous case to Q'. This time, remove all the leafs adjacent to vertex representing the door triangle. We can do the same analysis as in the previous case to find that if the number of leafs in Q' is d_1, then the number of degree three vertices is $d_3 = d_1 - 2$. But this time, one of the leafs is the door triangle, which is an equilateral triangle with side length m. Hence there are as many half edges as equilateral triangles. Hence, if m decreases, the MST-area of the chain decreases at least as much as the area of the chain.

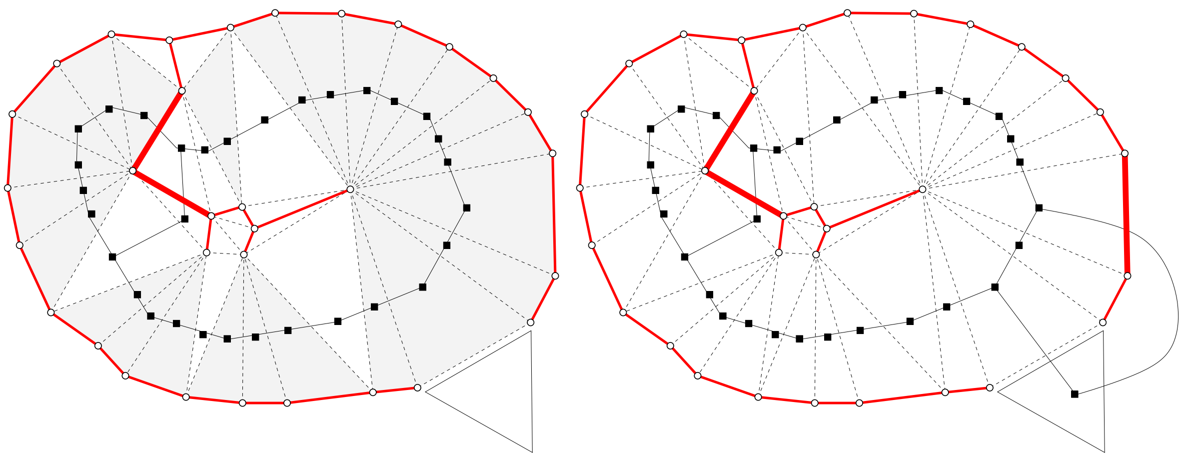

Figure 5: Two chains in a graph G plus the door triangle. During the process, the chain on the left appears when there are two maximal MST edges, whereas the one on the right appears when there are three maximal MST edges. The maximal edges are the thick MST edges.

5 Acknowledgements

I am grateful for interesting discussions with Andrea Clementi, Miriam di Ianni, Gianluca Rossi, Andreas Paffenholz, Ingo Schurr, and Emo Welzl.

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A Appendix: Analysis of ALG for $\alpha < 2$

Unfortunately, algorithm $\mathbf{ALG}$ does not have a constant approximation ratio for $\alpha < 2$. To see this, let us define analogous to Theorem 3

$$\mu(S, \alpha) := \sum_{i=1}^{|S|-1} \|e_i\|^\alpha.$$

Let us first show that $\mu(S, \alpha)$ cannot be bounded by a constant for $\alpha < 2$. Let $S_0, S_1, \ldots$ be the following set of instances. S_0 is the star instance with one station at the origin and six stations equidistantly distributed along the unit circle.

S_j can be obtained from S_{j-1} as follows. Consider the Delaunay triangulation of S_{j-1}. The instance S_j is obtained by adding a station at the midpoint of every edge of the triangulation. Note that the instances S_j are regular instances. That is, all edges of the Delaunay triangulation of S_j have the same length and all regions are equilateral triangles.

Let n_j and t_j be the number of stations and triangles of S_j. It holds $t_j = 6 \cdot 4^j$. We have seen earlier that if c is the number of edges of the convex hull of S_j, then $t_j = 2(n_j - 1) - c$. Therefore, the number of stations in S_j is $n_j = (t_j + c)/2 + 1 \geq t_j/2 + 1$. The length of the edges of the MST in S_j is 2^{-j}. Hence, we can bound

$$\mu(S_j, \alpha) = (n_j - 1) \cdot \left(2^{-j}\right)^\alpha \geq \frac{t_j}{2} \left(2^{-j}\right)^\alpha = 3 \left(\frac{4}{2^\alpha}\right)^j.$$

Obviously, $\mu(S_j, \alpha)$ tends to infinity for $\alpha < 2$ and $i \to \infty$. To see that the same bounds hold also for $\mathbf{ALG}$, we have to change S_j such that $\mathbf{ALG}$ has to pay for every edge of the MST of S_j. This can be done as follows. Consider the Delaunay triangulation of S_j. Now add two stations on every edge of the triangulation, each of them at distance ε from an endpoint of the edge. In this new instance, the MST will be composed of $n_j - 1$ edges of length $1 - 2\varepsilon$ and a number of edges of length ε. Every station will be incident to at most one edge of length $1 - 2\varepsilon$. Hence, all these edges all have to be payed, and the cost of $\mathbf{ALG}$ tends to $\mu(S_j, \alpha)$ for $\varepsilon \to \infty$. Noting $\mathrm{opt}(S_j) = 1$ for all j completes the proof.

B Appendix: Proof of Lemma 7

Proof of Lemma 7. Using Lemma 6, we can do an inductive proof on the number of triangles of the triangulation in the pocket. Consider a pocket with door e. For the base cases, assume the pocket is empty or an acute one. Note that all MST edges are not larger than one, since all points within the unit disc have distance at most one from the origin. Therefore, the door triangle of an empty pocket has side length at most one. This is also true for acute triangles, since the side length of the door triangle is bounded by the alrgest border edge, which is an MST edge as well.

Otherwise consider the pocket triangle incident to e, as shown in Figure 6. Let its other two edges be called a and b. Let w.l.o.g. $\|a\| \geq \|b\|$. This implies $\|e\| - \|a\| \leq \|b\|$ and $\beta \leq \frac{\pi}{4}$.

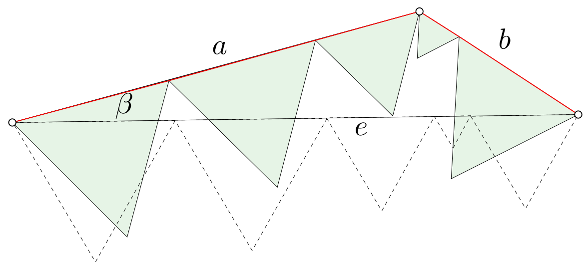

Figure 6:

For an edge e, let $PA(e)$ and $MST(e)$ be the pocket area and MST-area of the pocket defined by e. In Figure 6, the equilateral triangles along edge a and b represent the areas $MST(a) - PA(a)$ and $MST(b) - PA(b)$, respectively. The area of the triangles along e are obtained by copying the triangles along a and b, where the sides length of the later are shrunk by a factor of $(\|e\| - \|a\|)/\|b\|$. Note that the side length of the triangles along e add up to $\|e\|$, and the largest of these triangles is not larger than the largest triangle along a and b.

It remains to prove that $MST(e) - PA(e)$ is bounded by the triangles along e. Let us write $MST(a) - PA(a) := k_1\triangle(a)$ and $MST(b) - PA(b) := k_2\triangle(b)$ with $k_1, k_2 \leq 1$. The area of the triangles along e can then be written as $k_1\triangle(a) + k_2\triangle(e - a)$. Note that

$$MST(e) - PA(e) = MST(a) + MST(b) - PA(a) - PA(b) - \triangle(a, b, e).$$

In order to prove the lemma, we therefore need to show that $k_1\triangle(a) + k_2\triangle(b) - \triangle(a, b, e)$ is bounded by $k_1\triangle(a) + k_2\triangle(e - a)$.

$$\begin{aligned}
& k_1\triangle(a) + k_2\triangle(b) - \triangle(a, b, e) \\
= \; & k_1\triangle(a) + k_2\frac{\sqrt{3}}{4}b^2 - \frac{1}{2}ae\sin(\beta) \\
= \; & k_1\triangle(a) + k_2\frac{\sqrt{3}}{4}(e^2 + a^2 - 2ae\cos(\beta)) - \frac{1}{2}ae\sin(\beta) \\
= \; & k_1\triangle(a) + k_2\frac{\sqrt{3}}{4}(e^2 + a^2 - 2ae) + k_2\frac{\sqrt{3}}{4}(2ae - 2ae\cos(\beta)) - \frac{1}{2}ae\sin(\beta) \\
= \; & k_1\triangle(a) + k_2\triangle(e - a) + \frac{1}{2}ae(k_2\sqrt{3}(1 - \cos(\beta)) - \sin(\beta)) \\
\leq \; & k_1\triangle(a) + k_2\triangle(e - a)
\end{aligned}$$

The last inequality holds since $\sin(\beta) \geq \sqrt{3}(1 - \cos(\beta))$ in the range $0 \leq \beta \leq \frac{\pi}{3}$ and $k_2 \leq 1$. $\square$

C Appendix: Proof of Lemma 6

The process described in section 4 only works if all the pocket triangles of G are acute. For the general case, we need a slightly more complex process. We first need to introduce bunches.

Definition 10 *A* **bunch** *consists of a subset of pocket triangles obtained as follows. Consider the pocket triangles of $G(m)$. Now remove all the hypotenuses. The regions obtained in this way define the bunches: All pocket triangles that belong to the same region form a bunch.*

Obviously, the bunches partition the area of the pocket. Remember that we used the Delaunay triangulation to triangulate the point set. Therefore, the sum of opposite angles in the triangulation add up to at most π. It is therefore easy to see that a bunch always contains exactly one acute triangle. There are bunches that consist of a single triangle. This triangle must be acute. The acute triangle of a bunch is called **base**. We distinct between **border edges** and **inner edges** of a bunch. Note that inner edges are always the hypotenuse of some obtuse triangle.

Note that the adjacency graph of the triangles of a bunch is a tree. If the base is removed, we obtain at most three subtrees consisting in obtuse triangles only. We call them **obtuse trees**.

We are now ready to describe the process. It is basically the same process as in section 4. It is also described by the set of graphs $G(m)$, starting with $G(w_{\max})$ and ending with $G(w_{\min})$. In every stage, the maximal edges decrease simultaneously. The maximal edges in $G(m)$ always have length m.

A bunch changes in $G(m)$ if and only if its base has at least one maximal edge. As a reaction to the decrease of the maximal edges, all the inner edges of the bunch decrease as well.

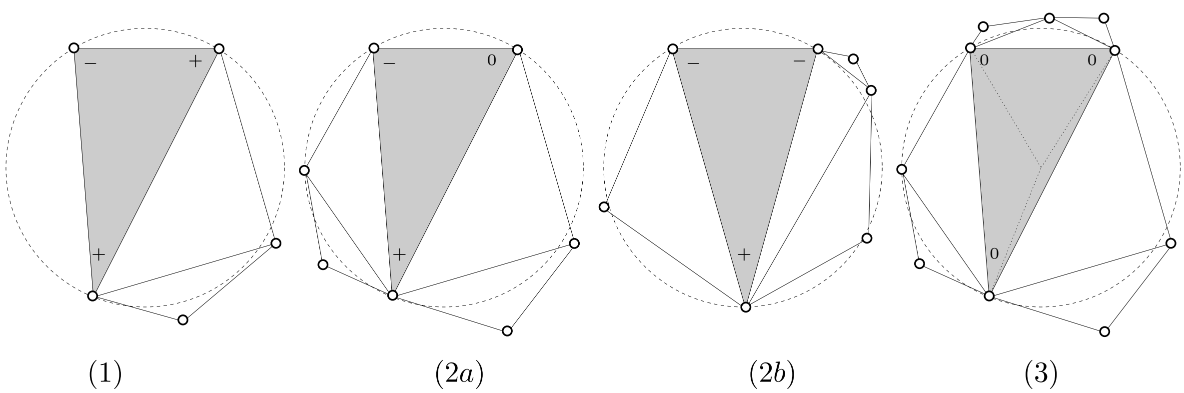

Figure 7: The four different types of base triangles. The $+$, $-$, and 0 signs indicate whether the corresponding angle increases, decreases or remains unchanged.

Figure 7 shows the different ways of how the edge lengths of a base can change. For the case distinction, we look at the number of base edges that cannot shrink. That is, the number of edges of the base that are neither maximal nor inner edges of the bunch. In case (1), the base has two non-shrinkable edges. Therefore only one base edge can be decreased and we can assume that it is maximal. The other two edge of the base remain constant. In case (2), only one edge of the base cannot shrink. If there are two maximal edges as shown in (2b), both of them need to shrink simultaneously. Otherwise, the non-maximal shrinkable edge reacts of the decrease of the maximal edge such that its opposite angle remains constant. In case (3), all base edges can shrink. This is done is such a way that all the angles are preserved.

To conclude the description of the process, we have to describe how the smaller edges of an obtuse triangle react if their hypothenuse shrinks. This can be done by the following two rules.

- Edges at the border of the bunch are not shrunk.
- The angle opposite of inner edges are preserved.

Not that whenever a obtuse triangle becomes acute during the process, its bunch gets split and the triangle becomes the base of a new bunch. At the end of the process, all triangles are acute and therefore form a bunch of their own. These are the three main properties of this process.

- All $G(m)$ have the Delaunay property, which is defined as follows. If α and β are opposite angles, it holds $\alpha + \beta \leq \pi$. This property holds for three reasons. First, it holds in G. Second, obtuse angles never increase during the process. And third, as long as α is obtuse, its opposite angle will not increase.
- During the process is that the MST of G is a valid MST in all $G(m)$. Note that during the process, only maximal edges and hypotenuses are decreased. Hypotenuses cannot be part of the MST, since choosing the other two edges of a obtuse triangle is always cheaper. Concerning the other edges, their ordering in terms of length remains unchanged during the process. Therefore the MST of G remains valid.
- In non-isosceles triangles, only the largest angle can decrease, whereas the other angles either increase or remain unchanged.
- The largest side of a triangle in G remains the largest side in all $G(m)$.

For the proof of the following lemma, we will use the following facts. If $\alpha + \delta \leq \pi$ and $\delta \geq \frac{\pi}{2}$, it holds

$$\cot(\delta) \leq \cot(\pi - \alpha) = -\cot(\alpha). \qquad (5)$$

For any angle α, it holds

$$sin(2\alpha) = 2\sin(\alpha)\cos(\alpha), \qquad 1 - \cos(\alpha) = 2\sin\left(\frac{\alpha}{2}\right)^2, \qquad \cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)}.$$

In a triangle with edge length a, b, and c as depicted in Figure 8 [15], we have

$$\sin(\gamma) = \frac{2\triangle(a,b,c)}{ab}, \qquad \cos(\gamma) = \frac{a^2+b^2-c^2}{2ab}, \qquad \cot(\gamma) = \frac{a^2+b^2-c^2}{4\triangle(a,b,c)}.$$

We will now prove that during the process, the areas of the bunches decrease monotonically. Let $A(m)$ be the area of a bunch in $G(m)$. In order to show that its area decreases when m decreases, we show

Lemma 11

$$\frac{\partial}{\partial x} A(x)\Big|_{x=m} \geq 0.$$

The fact that the derivation of $A(x)$ for $x = m$ is larger than zero not only means that $A(m)$ increases with growing m, it also means that it decreases for decreasing m.

The following derivations will be useful for our purposes. Consider a triangle with edges a, b, and c and a function $f(x)$ with $f(x_0) = c$. Then it holds

$$\begin{aligned}
& \frac{\partial}{\partial x}\triangle(a,b,f(x))\Big|_{x=x_0} \\
= \;& \frac{1}{2}\,\frac{\left(\frac{\partial}{\partial x}f(x)\right) f(x) \left(a^2+b^2-(f(x))^2\right)}{\triangle(a,b,f(x))}\Bigg|_{x=x_0} \\
= \;& \frac{1}{2}\,\frac{\left(\frac{\partial}{\partial x}f(x)\right) f(x) 2ab\cos(\gamma)}{4\cdot\triangle(a,b,f(x))}\Bigg|_{x=x_0} \\
= \;& \frac{1}{2}\,\frac{\left(\frac{\partial}{\partial x}f(x)\right) f(x) 2ab\cos(\gamma)}{4\cdot\frac{1}{2}ab\sin(\gamma)}\Bigg|_{x=x_0} \\
= \;& \frac{1}{2}\left(\frac{\partial}{\partial x}f(x)\right) f(x)\cot(\gamma)\Bigg|_{x=x_0}
\end{aligned}$$

If we have $f(x) = \sqrt{g(x)}$, then

$$\frac{\partial}{\partial x}\triangle(a,b,f(x))\Big|_{x=x_0} = \frac{1}{2}\left(\frac{\partial}{\partial x}f(x)\right) f(x)\cot(\gamma)\Big|_{x=x_0} = \frac{1}{2}\,\frac{1}{2}\,\frac{\partial}{\partial x}g(x)\cot(\gamma)\Big|_{x=x_0}$$

Hence, for $f(x) = x$ and $x_0 = c$ we get

$$\frac{\partial}{\partial x}\triangle(a,b,x)\Big|_{x=c} = \frac{1}{2}c\cot(\gamma).$$

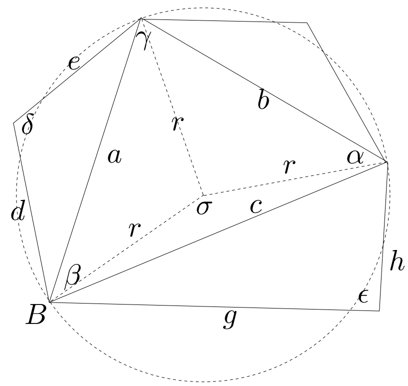

Figure 8: This figure shows a base triangle with edges a, b and c. The radius of its circum circle is denoted by r.

For $f(x) = cx$ and $x_0 = 1$ we get

$$\frac{\partial}{\partial x} \triangle(a, b, cx)\Big|_{x=1} = \frac{1}{2}c^2 \cot(\gamma).$$

If all edges of a triangle we decreased by a factor x, we get

$$\left(\frac{\partial}{\partial x} \triangle(ax, bx, cx)\right)\Big|_{x=1} = \left(\frac{\partial}{\partial x}\frac{1}{2}ax \cdot cx \sin(\beta)\right)\Big|_{x=1} = ac\sin(\beta). \tag{6}$$

If edge a and angle β remain constant, we get

$$\left(\frac{\partial}{\partial x} \triangle(a, b(x), x)\right)\Big|_{x=c} = \left(\frac{\partial}{\partial x}\frac{1}{2}a \cdot x \sin(\beta)\right)\Big|_{x=c} = \frac{1}{2}a\sin(\beta). \tag{7}$$

Proof of Lemma 11. We will prove this lemma by induction. For the base cases of the induction, we consider four cases from Figure 7, where all the obtuse trees consist of a single triangle.

Let us start with case (1). Consider a bunch consisting of the two triangles incident to c in Figure 8. Assume c is a maximal edge. Hence, edge c gets decreased. We then can obtain

$$\begin{aligned}
\frac{\partial}{\partial x} A(x)\Big|_{x=m} &= \left(\frac{\partial}{\partial x} \triangle(a, b, x) + \triangle(x, g, h)\right)\Big|_{x=c} \\
&= \frac{1}{2}c\cot(\gamma) - \frac{1}{2}c\cot(\varepsilon) \\
&\geq \frac{1}{2}c\cot(\gamma) - \frac{1}{2}c\cot(\pi - \gamma) = 0.
\end{aligned}$$

Here we used $\varepsilon \geq \frac{\pi}{2}$ and that because of the Delaunay property we have $\varepsilon + \gamma \leq \pi$, which allows to use Equation (5). For the case (2a), consider a bunch composed by the three triangles incident to B of Figure 8. We assume that only c is maximal. This means that b and α remain constant. Hence, a becomes $a(c) = \sqrt{b^2 + c^2 - 2bc\cos(\alpha)}$.

$$
\begin{aligned}
\frac{\partial}{\partial x} A(x)\Big|_{x=m} &= \frac{\partial}{\partial x} \left(\triangle(a(x), b, x) + \triangle(a(x), d, e) + \triangle(x, g, h)\right)\Big|_{x=c} \\
&= \frac{\partial}{\partial x} \left(\frac{1}{2} sin(\alpha) bx + \triangle(a(x), d, e) + \triangle(x, g, h)\right)\Big|_{x=c} \\
&= \frac{1}{2} sin(\alpha) b + \left(\frac{\partial}{\partial x} a(x)\right) a(x)\Big|_{x=c} \cot(\delta) + \frac{1}{2} c \cot(\varepsilon) \\
&\geq \frac{1}{2} \sin(\alpha) b + \frac{1}{2} \left(c - b\cos(\alpha)\right) \cot(\pi - \alpha) + \frac{1}{2} c \cot(\pi - \gamma) \\
&= \frac{1}{2} \sin(\alpha) b - \frac{1}{2} \left(c - b\cos(\alpha)\right) \cot(\alpha) - \frac{1}{2} c \cot(\gamma) \\
&= \frac{1}{16} \frac{16\,\triangle(a, b, c)^2 + c^4 + b^4 + a^4 - 2\,b^2a^2 - 2\,c^2b^2 - 2\,c^2a^2}{c\triangle(a, b, c)} \\
&= \frac{1}{16} \frac{16\,\triangle(a, b, c)^2 - 16\triangle(a, b, c)^2}{c\triangle(a, b, c)} = 0
\end{aligned}
$$

We again used Equation (5) for get the inequality here. For case (3), consider a bunch containing all four triangles of Figure 8. We partition the bunch into three parts by splitting the base into three subtriangles using the midpoint of the circum cycle. Let r be the radius of the cycle. We now prove that the total area of each of these subtriangles together with its corresponding obtuse triangle decreases if all sides of the base triangle are decreased. Note that here we assume that the edges get decreased by a multiplicative factor x, starting at $x = 1$. This can be done since we are just interested in the sign of the derivation of $A(m)$.

$$\begin{aligned}
\frac{\partial}{\partial x}A(x)\Big|_{x=1} &= \left(\frac{\partial}{\partial x}\triangle(rx,rx,cx)+\triangle(cx,g,h)\right)\Big|_{x=1} \\
&= r^2\sin(\sigma)+\frac{1}{2}c^2\cot(\varepsilon) \\
&\geq r^2\sin(\sigma)+\frac{1}{2}c^2\cot(\pi-\gamma) \\
&= r^2\sin(\sigma)+\frac{1}{2}c^2\cot(\pi-\frac{\sigma}{2}) \qquad (8)\\
&= r^2\sin(\sigma)+\frac{1}{2}\left(r^2+r^2-2r^2\cos(\sigma)\right)\cot(\pi-\frac{\sigma}{2}) \\
&= r^2\left(\sin(\sigma)-2\sin(\frac{\sigma}{2})^2\cot(\frac{\sigma}{2})\right) \\
&= r^2\left(\sin(\sigma)-2\sin(\frac{\sigma}{2})\cos(\frac{\sigma}{2})\right)=0
\end{aligned}$$

For (8), we used that σ is a central angle of γ and therefore $\sigma = 2\gamma$ [15]. As for case (2b), one can give a similar proof. But this case can also be handled by case (2a), since we can assume that the two maximal sides take turns in decreasing by infinitesimal steps. This completes the base cases of our inductive proof.

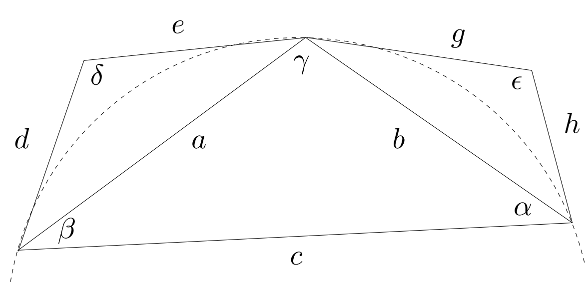

Figure 9: The inductive step.

For the inductive step, consider a bunch B in $G(m)$. We call the obtuse triangles that have two edges that are border edges of B **leaf triangles**. We consider two cases.

For the first case, let t be a triangle that has two leaf triangles l_1 and l_2. Let B' be the bunch obtained if we remove l_1 and l_2 in $G(m)$. We know by induction that the area of B' decreases in $G(m)$. In order to make the inductive step, we prove that the increase of the total area of the triangles t, l_1, and l_2 in B is bounded by the increase of t in B'. Consider Figure 9. Let us assume that the edge c gets decreased by decreasing a multiplicative factor x starting from 1. The increase of t, l_1, and l_2 in B can be written as

$$\frac{\partial}{\partial x}\left(\triangle(ax,bx,cx)+\triangle(ax,d,e)+\triangle(bx,g,h)\right)\Big|_{x=1}$$

wheras the increase of t in B is

$$\frac{\partial}{\partial x}\left(\triangle(a,b,cx)\right)\Big|_{x=1}$$

We now get

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\triangle(ax,bx,cx)+\triangle(ax,d,e)+\triangle(bx,g,h)-\triangle(a,b,cx)\right)\Big|_{x=1}\\
=\quad & sin(\beta)ac+\frac{1}{2}a^2\cot(\delta)+\frac{1}{2}b^2\cot(\varepsilon)-\frac{1}{2}c^2\cot(\gamma)\\
\geq\quad & \sin(\beta)ac+\frac{1}{2}a^2\cot(\pi-\alpha)+\frac{1}{2}b^2\cot(\pi-\beta)-\frac{1}{2}c^2\cot(\gamma)\\
=\quad & \sin(\beta)ac-\frac{1}{2}a^2\cot(\alpha)-\frac{1}{2}b^2\cot(\beta)-\frac{1}{2}c^2\cot(\gamma)\\
=\quad & 2\triangle(a,b,c)-\frac{16\cdot\triangle(a,b,c)^2}{8\cdot\triangle(a,b,c)}=0
\end{aligned}$$

In the second case of the induction step, we assume that there is only one leaf triangle l_1. Using similar arguments, we have to prove

$$\frac{\partial}{\partial x}\left(\triangle(a(x),b,x)+\triangle(a(x),d,e)-\triangle(a,b,x)\right)\Big|_{x=c}\geq 0 \tag{9}$$

with $a(x)=\sqrt{b^2+x^2-2bx\cos(\alpha)}$. This is very similar to (8). Note that we bounded the last term there by $-c\cot(\gamma)/2$, which is exactly equal to $-\triangle(a,b,x)$. This completes the proof of Lemma 11 □

So far, we just looked at the pocket area during the process and proved that is monotonically decreases. But what we actually need is to show that the net-area decreases. Hence, we have to show that in each $G(m)$, the decrease of the pocket area is at least as large as the decrease of the MST-area. To do this, we will partition the bunches of $G(m)$ into so-called **chains** for which we will prove that their total net-area decreases monotonically.

Consider a graph $G(m)$ for fixed m. Chains are defined in terms of a graph Q. The vertex set of Q are the bases of $G(m)$ plus the door triangle. Each maximal MST edge e of $G(m)$ creates the following set of edges in Q. If we remove e, the MST is divided into two subtrees. Let $R(e)$ be the ring of triangles that separates the two subtrees. For any pair of adjacent triangles in $R(e)$, we add an edge in Q. Note that all triangles of $R(e)$ are isosceles in $G(m)$ and are therefore bases. Note that if e is a border edge, the door triangle is also part of the ring and it is connected with the triangle incident to e and the pocket triangle incident to the door, as shown on the right of Figure 5. This completes the definition of the graph Q.

The **chains** are defined as the connected components of Q. Let Q' be a chain. We can define the area, the MST-area, and the net-area of a chain as follows. The **area of a chain** is equal to the sum of the areas of all the bunches whose base belongs to Q'. Concerning the MST-area, note that every MST half edge h of $G(m)$ and its corresponding MST-triangle can be assigned to a unique chain, namely the one that contains the unique triangle h is incident to. The **MST area of a chain** is equal to the sum of the areas of all MST triangles belonging to the chain. The **net-area of a chain** is its area minus its MST-area. Since the net-area of a pocket is equal to the sum of the net-areas of all its chains, all we have to do to complete the proof is to show that the net-area of a chain decreases monotonically.

Remember that in $G(m)$, only the maximal MST edge decrease. Therefore only the MST-triangles of maximal MST edges decrease. In what follows, we will show that the decrease of the area of the chain makes up for the decrease of these **maximal MST-triangles**.

Some chains contain only a single triangle and no maximal MST-triangles. For these chains, it is obvious that the net-area decreases. Let us now look at chains that contain maximal MST-triangles. We need to consider two cases.

In case (i), we assume that the door triangle is not a vertex of Q'. The **border of** Q' is the cycle in G with smallest area that contains all the base triangles belonging to Q'. The base triangles of Q' that are incident to a border edge are called **border triangles**. On the left of Figure 5, the border triangles are shaded. In every border triangle, we call the angle opposite to the border edge **inner angle**.

Lemma 12 *Let $b(m)$ be the area of all bunches whose bases are a border triangle of Q' in $G(m)$. Then*

$$\frac{\partial}{\partial x} b(m)\bigg|_{x=m} \geq \sqrt{3}m$$

Proof. As a first step, consider the border triangles of Q' in G. In is easy to see that the inner angles of these triangles sum up to 2π in G.

Now look at a border triangle t. Let $u(m)$ be the area of the bunch containing t in $G(m)$. In $G(m)$, t has at least two edges of length m and one edge of length $a \leq m$, say. The only edge that can have an obtuse tree attached is a. If there is one, we denote its area in $G(m)$ by $o(m)$. Let $\alpha(m)$ and α be the inner angle of t in $G(m)$ and G, respectively. In order to prove the lemma, we show

$$\frac{\partial}{\partial x} u(x)\bigg|_{x=m} \geq \frac{\sqrt{3}m\alpha}{2\pi}.$$

- If $\frac{\pi}{2} \leq \alpha \leq \pi$, t is an equilateral triangle with side length m in $G(m)$, forming a bunch of its own. This is why: Note that in G, edge a was the largest edge of t. Since this property is preserved in all $G(m)$ we have $a \geq m$ in $G(m)$, but clearly it also holds $a \leq m$. We additionally know that t forms a bunch of its own in $G(m)$, since only along a, there could be an obtuse tree. But the angle opposite of α can be at most $\frac{\pi}{2}$ in G.

Therefore

$$\frac{\partial}{\partial x}u(x)\bigg|_{x=m} = \frac{1}{2}\sqrt{3}m = \frac{\sqrt{3}m\pi}{2\pi} \geq \frac{\sqrt{3}m\alpha}{2\pi}.$$

- If $\frac{\pi}{3} \leq \alpha < \frac{\pi}{2}$, t is an equilateral triangle with side length m in $G(m)$, but it might not be a bunch of its own. From case (3) of Lemma 11, we know that the area of one third of t together with the obtuse tree o decreases. The decrease of the remaining two thirds of t is enough to get desired bound.

$$\frac{\partial}{\partial x}u(x)\bigg|_{x=m} = \frac{\partial}{\partial x}\left(\frac{2}{3}t(x) + \frac{1}{3}t(x) + o(x)\right)\bigg|_{x=m} \geq \frac{2}{3}\frac{1}{2}\sqrt{3}m \geq \frac{\sqrt{3}m\alpha}{2\pi}.$$

- If $\alpha < \frac{\pi}{3}$, it holds $\alpha(m) \geq \alpha$. If $u(m)$ is a bunch of its own, it holds

$$\begin{aligned}
\frac{\partial}{\partial x}u(x)\bigg|_{x=m} &= \left(\frac{\partial}{\partial x}\triangle(x,x,a)\right)\bigg|_{x=m} = \frac{a^2 m}{4\cdot\triangle(m,m,a)} = \frac{m(m^2+m^2-2m^2\cos(\alpha(m)))}{2\sin(\alpha(m))m^2} \\
&= \frac{m(1-\cos(\alpha(m)))}{\sin(\alpha(m))} = \frac{m(\cos(\alpha(m)))}{1+\sin(\alpha(m))} \\
&\geq \frac{m(\cos(\alpha))}{1+\sin(\alpha)} \geq \frac{\sqrt{3}m\alpha}{2\pi}. \qquad (10)
\end{aligned}$$

The last inequality is proved by Figure 10. If $u(m)$ is not a bunch of its own, all sides of t are decreased such that the angles are preserved. Again we can partition t into three subtriangles using the middle point of the circum circle. We already know from case (3) of Lemma 11 that the area of the subtringle incident to a, together with the obtuse tree attached to it, decreases. The total area of the other two parts is at least one half of the triangle, and we obtain

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\frac{1}{2}u(x)\right)\bigg|_{x=m} &= \left(\frac{\partial}{\partial x}\frac{1}{2}\triangle(x,x,a(x)) + \frac{1}{2}\triangle(x,x,a(x)) + o(x)\right)\bigg|_{x=m} \\
&\geq \frac{1}{2}\sin(\alpha(m))m + 0 \geq \frac{1}{2}\sin(\alpha)m \geq \frac{\sqrt{3}m\alpha}{2\pi}.
\end{aligned}$$

The last inequality is proved by Figure 10.

□

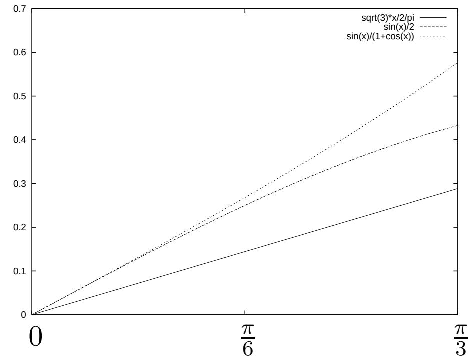

Figure 10: Three functions appearing in the proof of Lemma 12.

Let us now change Q' slightly. Let e be a maximal MST edge belonging to Q'. Assume its two incident triangles are q_1 and q_2. Add two new vertices h_1 and h_2 to Q'. They represent the half edges of e. Then remove the edge q_1q_2 from Q' and add q_1h_1 and q_2h_2.

If we do this for all maximal MST edges, Q' becomes a tree. Let d_i be the number of vertices of Q' with degree i, let n and e be the number of vertices and edges, respectively. From $e = n - 1$, $n = d_1 + d_2 + d_3$ and $2e = 3d_3 + 2d_2 + d_1$ we get $d_3 = d_1 - 2$.

The vertices of degree three are maximal equilateral triangles in $G(m)$, forming a bunch of their own. Let D_3 be the set of this triangles. The vertices of degree one are the maximal MST half edges in Q'.

We are now ready to show that the net-area of Q' decreases. Let $Q'(m)$ be the net-area of Q' in $G(m)$. Let $K(m)$ be the area of the bunches of Q' which are not considered in $b(x)$ and D_3. Remember that for all the bunches in $K(m)$, Lemma 11 applies. Let M be the MST-area generated by non-maximal MST edges. Then

$$\begin{aligned}\left.\frac{\partial}{\partial x}Q'(x)\right|_{x=m} &= \left.\frac{\partial}{\partial x}\left(d_3 \cdot \triangle(x) + b(x) + K(x) - d_1 \cdot \triangle(x) - M\right)\right|_{x=m} \\ &\geq d_3\frac{1}{2}\sqrt{3}m + \sqrt{3}m + 0 + -d_1\frac{1}{2}\sqrt{3}m + 0 = 0.\end{aligned}$$

For case (ii), assume that the door triangle belongs to Q'. Apply the edge splitting of the previous case to Q'. This time, remove all the leafs adjacent to vertex representing the door triangle. We can do the same analysis as in the previous case to find that if the number of leafs in Q' is d_1, then the number of degree three vertices is $d_3 = d_1 - 2$. But this time, one of the leafs is the door triangle, which is an equilateral triangle. Hence there are as many half edges as equilateral triangles. Hence, if m decreases, the MST-area decreases at least as much as the maximal MST triangles. Since all the remaining bunches decrease as well, we are done.