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The existence of superinvolutions

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Abstract

Superinvolutions on graded associative algebras constitute a source of Lie and Jordan superalgebras. Graded versions of the classical Albert and Albert–Riehm Theorems on the existence of superinvolutions are proven. Surprisingly, the existence of superinvolutions of the first kind is a rare phenomenon, as non-trivial central division superalgebras are never endowed with this kind of superinvolutions.

Keywords: Superinvolution; Albert Theorem; Albert–Riehm Theorem; Superalgebra; Simple; Central

1. Introduction

Albert's Theorem (see [1,12]) asserts that a finite dimensional central simple algebra has an involution of the first kind if and only if the order of its class in the Brauer group is at most 2, while Albert–Riehm Theorem (see [11,12]) asserts that a necessary and sufficient condition for the existence of an involution of the second kind is that the so-called corestriction is trivial in the Brauer group.

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Given an associative superalgebra (that is, a graded associative algebra) $A = A_0 \oplus A_1$, a superinvolution is a graded linear map $\xi : A \to A$, which is a superantiautomorphism (that is, $\xi(xy) = (-1)^{xy}\xi(y)\xi(x)$ for any homogeneous elements $x, y \in A$) and such that ξ^2 is the identity map. The set of skew elements of a superinvolution $\{x \in A\colon\ \xi(x) = -x\}$ is a Lie superalgebra under the graded bracket: $[x, y] = xy - (-1)^{xy}yx$, while the set of fixed elements $\{x \in A\colon\ \xi(x) = x\}$ is a Jordan superalgebra under the supersymmetrized product: $x \circ y = xy + (-1)^{xy}yx$. Many of the classical simple Lie and Jordan superalgebras (see [6,10]) arise in this way. Superinvolutions on primitive associative superalgebras were studied by Racine [9] and used in the classification of the simple Jordan superalgebras [10].

The purpose of this paper is to obtain necessary and sufficient conditions for a finite dimensional central simple associative superalgebra to be endowed with a superinvolution, thus obtaining analogues in the graded setting of the classical Albert and Albert–Riehm Theorems.

Surprisingly, superinvolutions of the first kind are more difficult to deal with, and it turns out that the natural analogue of Albert Theorem is false: even if the class in the Brauer–Wall group of a central simple algebra has order at most 2, the superalgebra may have no superinvolution of the first kind. Indeed, central simple superalgebras of odd type never have superinvolutions of the first kind (Theorem 28), and the same happens for central division superalgebras of even type with nontrivial odd part (Lemma 31 and Theorem 32). However, Albert Theorem has a graded version, where superinvolutions are substituted by superantiautomorphisms whose square is the grading automorphism (Theorem 38).

The situation is nicer for superinvolutions of the second kind. For these, the natural graded version of the classical Albert–Riehm Theorem holds.

The paper is organized as follows. The next section is intended to recall the basic definitions and results on associative superalgebras. The basic sources are Lam's book [8] and Racine's paper [9]. Some results will be proved in slightly different ways, useful for our purposes. Then Section 3 will deal with superinvolutions of the first kind on central simple superalgebras. It will be shown that the existence of such superinvolutions is severely restricted. Section 4 will be devoted to prove the above mentioned graded version of the classical Albert Theorem, while superinvolutions of the second kind and the graded version of Albert–Riehm Theorem will be the object of the last section.

When this work was complete, the authors learnt from Professor Michel Racine that his former student Amer Jaber [3–5] had obtained similar results, although the approach and the proofs are different. Actually, [3] deals with the existence of superinvolutions of the first kind, which is the content of Section 3 in this paper, [4] with superinvolutions of the second kind (Section 5 here), and [5] with 'pseudo-superinvolutions,' which are the superantiautomorphisms whose square is the grading automorphism (Section 4).

2. Basic concepts

2.1. *Definitions and notations*

We recall some basic definitions (compare with [8, Chapter IV]). Let F be a field of characteristic different from 2. (*This restriction will be assumed without mention throughout the paper*.) A *superalgebra* (also called *graded algebra*) A is an (associative) F-algebra given in the form $A = A_0 \oplus A_1$, $F = F \cdot 1 \subseteq A_0$ and $A_iA_j \subseteq A_{i+j}$ (subscripts modulo 2). We observe that A_0 is a subalgebra of A. The elements of $hA := A_0 \cup A_1$ are called *homogeneous elements* of A.

For $h \in hA - \{0\}$ we define *the degree* δh of h by $\delta h = i$ if $h \in A_i$ $(i = 0, 1)$. To simplify the notation, for every homogeneous element x we define $(-1)^x := (-1)^{\delta x}$.

A subspace $S \subseteq A$ is called *graded* if it is the direct sum of the intersections $S_i := S \cap A_i$. We define $h(S) = S \cap h(A)$. The *graded center* of the superalgebra A is $\widehat{Z}(A) := \{x \in hA \mid xh = (-1)^{xh} hx\ \forall h \in hA\}$. We shall call A a *central superalgebra* over F if $\widehat{Z}(A) = F$. The superalgebra A is said to be a *simple superalgebra* over F if A has no proper ($\neq 0, \neq A$) graded twosided ideals. A finite dimensional simple superalgebra which is also a central superalgebra is called *central simple superalgebra* (CSS, CSGA in Lam's book, see [8]).

The ordinary center of A, i.e. $Z(A) = \{x \in A \mid xa = ax\ \forall a \in A\}$, is a graded subalgebra. If A is a CSS over F, then $Z(A) = F \oplus Z_1$ $(Z_1 \subseteq A_1)$. If $Z_1 = 0$, we say that A is of *even type*. If $Z_1 \neq 0$, we say that A is of *odd type*.

We denote the *graded tensor product* of two graded algebras A and B by $A \mathbin{\widehat{\otimes}} B$. We denote the Brauer group of a field F with $B(F)$ (respectively the Brauer–Wall group with $BW(F)$). The opposite (respectively superopposite) algebra of a central simple algebra (respectively CSS) A is denoted A^{op} (respectively A^s; in Lam's book the algebra A^s is denoted by A^*, see [8, pp. 80, 99]).

Examples 1. We list some important examples of CSSs.

(a) We denote with (A) the algebra A with the grading given by $A_1 = 0$.
(b) Let A, B be CSSs. Then $A \mathbin{\widehat{\otimes}} B$ is a CSS.
(c) Let $a \in F^\times$. The space $F \oplus Fu$ with the relation $u^2 = a$ and the grading $\delta u = 1$ defines a CSS denoted by $F\langle\sqrt{a}\,\rangle$. We call it *quadratic graded algebra*.
(d) For $a, b \in F^\times$ we define the *graded quaternion algebra* as follows: $\langle a, b\rangle := F\langle\sqrt{a}\,\rangle \mathbin{\widehat{\otimes}} F\langle\sqrt{b}\,\rangle$ (see [8, p. 87]).
(e) Let D be a central division algebra over a field F. The algebra of $(n+m) \times (n+m)$-matrices $M_{n+m}(D)$ can be viewed as CSS by taking the diagonal components $M_n(D)$ and $M_m(D)$ as the even part and the off-diagonal components as the odd part.

Remark 2. We observe that the definitions in examples (c) and (d) cover all nontrivial gradings on a quadratic algebra and on a quaternion algebra. For example, given a quaternion algebra with nontrivial grading, the odd elements are orthogonal to 1 with respect to the (quaternionic) norm and hence they have zero trace. Since the restriction of the norm to the space of odd elements is not degenerate, then the algebra is forced to be of the form $\langle a, b\rangle$ for some $a, b \in F^\times$.

For completeness, we recall the structure theorems for CSSs (see [8, Chapter IV, 3.6 and 3.8]).

Theorem 3. *Let A be a CSS of odd type. Then*:

(1) $Z(A) = F \oplus Fz$*, where* $z \in Z_1$ *and* $z^2 = a \in F^\times$*. The square class of* a *does not depend on the choice of* $z \in Z_1 - \{0\}$*, and* $Z(A) \simeq F\langle\sqrt{a}\,\rangle$ *as graded algebras.*
(2) *There are graded algebra isomorphisms*

$$A \simeq (A_0) \mathbin{\widehat{\otimes}} F\langle\sqrt{a}\,\rangle \simeq (A_0) \otimes F\langle\sqrt{a}\,\rangle.$$

(3) *If* $a \notin F^{\times 2}$*, then* A *is a central simple algebra over* $Z(A) \simeq F(\sqrt{a})$*. If* $a \in F^{\times 2}$*, then* $Z(A) \simeq F \times F$*, and* $A \simeq A_0 \times A_0$ *(as ungraded algebra).*

In any case, A is a semisimple separable F-algebra.

Theorem 4. *Let A be a CSS of even type, $A_1 \neq 0$. Suppose A is isomorphic, as ungraded algebra, to $M_n(D)$, the central simple F-algebra of $n \times n$-matrices with entries in the central division algebra D over F. Then*:

(1) *There exists an element $z \in Z(A_0)$ such that $Z(A_0) = F \oplus Fz$ and $z^2 = a \in F^\times$. The element z is determined up to a scalar multiple by these properties, and hence the square class of a is uniquely determined.*
(2) *Suppose $a \in F^{\times 2}$. Then $Z(A_0) \simeq F \times F$ and $A \simeq M_{r+s}(F) \widehat{\otimes} (D)$ with $r+s=n$. Moreover, $A_0 \simeq M_r(D) \times M_s(D)$.*
(3) *Suppose $a \notin F^{\times 2}$, and the field $Z(A_0) \simeq F(\sqrt{a})$ can be embedded into D. Then there exists a grading on D such that $A \simeq (M_n(F)) \widehat{\otimes} D$. In this case, $A_0 \simeq M_n(D_0)$ is a central simple algebra over $Z(A_0)$.*
(4) *Suppose $a \notin F^{\times 2}$, and the field $Z(A_0) \simeq F(\sqrt{a})$ cannot be embedded into D. Then $n = 2m$ is even, and $A \simeq (M_m(D)) \widehat{\otimes} \langle -a, 1\rangle$ as graded algebras. In this case, $A_0 \simeq M_m(D) \widehat{\otimes} F(\sqrt{a})$ is a central simple algebra over $Z(A_0)$.*

In any case, A_0 is a semisimple separable F-algebra.

We call an algebra *central division superalgebra* (CDS) if it is a CSS where every nonzero homogeneous element is invertible.

Remark 5. If A is a CSS of even type ($A_1 \neq 0$, $Z(A_0) = F \oplus Fz$ and $z^2 = a \in F^\times$) with $a \notin F^{\times 2}$ (i.e. in the cases (3) and (4) of the theorem above), then we may write it as

$$A \simeq \big(M_n(F)\big) \widehat{\otimes} \Delta$$

where Δ is a CDS (see for example Theorem 2 in [2] or the proof of [8, Chapter IV, 3.8]).

On the other hand, if A is a CSS of odd type, then it is of the form $(M_k(F)) \widehat{\otimes} \Delta$, where Δ is a CDS of odd type.

Let $A = A_0 + A_1$ be a superalgebra. For all $x \in A_0$, $y \in A_1$ we define $\nu(x+y) = x - y$. The induced map ν is a graded automorphism of A called the *grading automorphism* (the main involution in Lam's book, see [8, Chapter IV, Definition 3.7]). If A is a CSS of even type with $Z(A_0) = F1 + Fz$ and $z^2 \in F^\times$, then recall that $\nu(x) = zxz^{-1}$; in particular, we have $uz = -zu$ for all $u \in A_1$. A *superantiautomorphism* of a superalgebra A is a graded additive map $\sigma : A \to A$ such that for all $a_\alpha \in A_\alpha$ and $b_\beta \in A_\beta$ $\sigma(a_\alpha b_\beta) = (-1)^{\alpha\beta}\sigma(b_\beta)\sigma(a_\alpha)$. We call *superinvolution* of A a superantiautomorphism τ such that $\tau^2(x) = x$ for all $x \in A$. As for involutions, we say that the superinvolution is *of the first kind* if it is F-linear, and *of the second kind* otherwise.

Remark 6. Let A, B be CSSs over F with superinvolutions τ_A and τ_B. The map $\tau_A \otimes \tau_B$ is a superinvolution on $A \widehat{\otimes} B$.

Before the study of involutions of the first kind, we consider the case of the CSS $M_{n+m}(F)$: it shows that the existence of a superantiautomorphism does not always imply the existence of a superinvolution. However, here it is easy to see that there is always a superantiautomorphism whose square is the grading automorphism.

Proposition 7. *The CSS $M_{n+m}(F)$ has always a superantiautomorphism φ with $\varphi^2 = \nu$. It has a superinvolution of the first kind if and only if $n = m$ or nm is even.*

Proof. As usual, we denote the transpose of a matrix a with a^t. We observe that the map

$$\varphi : M_{n+m}(F) \to M_{n+m}(F), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix}$$

is a superantiautomorphism. Moreover, one can check easily that $\varphi^2 = \nu$.

For the case $n = m$, it is enough to observe that the map

$$\tau : M_{n+n}(F) \to M_{n+n}(F), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}$$

is a superinvolution. If $n \neq m$ and τ is a superinvolution on $M_{n+m}(F)$, then τ is adjoint to a superform (see [9, Theorem 7] or Theorem 21). Such a superform can be defined if and only if n or m is even. $\square$

2.2. *The graded Skolem–Noether Theorem*

In this section we give a version of the Skolem–Noether Theorem for superalgebras (see also Lemma 1 in [11]).

For any homogeneous and invertible element of a superalgebra A, consider the *inner automorphism* ι_a given by

$$\iota_a(x) = (-1)^{ax} a x a^{-1}$$

for any homogeneous $x \in A$. We denote the disjoint union with $\sqcup$. In the next proposition we describe the set $\mathrm{Autg}(A)$ of graded F-linear automorphisms of A.

Proposition 8. *Let A be a CSS. Then* $\mathrm{Autg}(A) = \iota_{A_0^\times} \sqcup \iota_{A_1^\times}$.

Proof. Let φ be an element of $\mathrm{Autg}(A)$. We can write $A = \mathrm{End}({}_\Delta V)$, where V is a graded vector space with the action of the CDS Δ on the left. We assume that $A_1 \neq 0$ (i.e. $V_1 \neq 0$). Let $\Psi : \Delta^s \mathbin{\widehat{\otimes}} A \to \mathrm{End}_F(V)$, defined by $d \otimes a \mapsto (\Psi(d \otimes a) : v \mapsto (-1)^{dv} dva)$. By dimension count, Ψ is an isomorphism of CSS. There are two $\Delta^s \mathbin{\widehat{\otimes}} A$-module structures on V, namely $v.(d \otimes a) = (-1)^{dv} dva$ and $v \diamond (d \otimes a) = (-1)^{dv} dva^{\varphi}$. But the only irreducible $\mathrm{End}_F(V)$-modules of $\mathrm{End}_F(V)$ are V and V^s, the superopposite module of V, i.e. V with the roles of V_0 and V_1 interchanged.

Two cases may occur.

1. There exists an F-linear even isomorphism $\sigma : V \to V$ such that $v.(d \otimes a)\sigma = (v\sigma) \diamond (d \otimes a)$ for all $d \in \Delta$, $v \in V$, $a \in A$.

Setting $a = 1$ we obtain that $\sigma \in A_0$. Then, setting $d = 1$, we get $\varphi = \iota_\sigma$.

2. There exists an F-linear odd isomorphism $\sigma : V \to V$ such that

$$v.(d \otimes a)\sigma = (-1)^a (-1)^d (v\sigma) \diamond (a \otimes d)$$

for all $d \in \Delta$, $v \in V$, $a \in A$. Setting again, as before, $a = 1$ and $d = 1$ we get $\varphi = \iota_\sigma$. $\square$

Remark 9. If, in the proposition above, A is even, then the grading automorphism is $\nu = \iota_z$, where z is any element such that $Z(A_0) = F1 + Fz$ and $z^2 \in F^\times$. If A is odd, then $Z(A) = F1 + Fz$ for an invertible odd element z, and again $\nu = \iota_z$. Since here $A_1 = A_0 z$, it follows that $\iota_{A_1^\times} = \nu \circ \iota_{A_0^\times}$.

Remark 10. If $\varphi \in \operatorname{Autg}(A)$ fixes $Z(A)$ and $Z(A_0)$ elementwise, then $\varphi \in \iota_{A_0^\times}$.

2.3. The graded Jacobson density theorem and superinvolutions

The results of Michel Racine (see [9]) for prime superalgebras with superinvolution will be proven in a slightly different way. In this context we use words prime and semiprime in the obvious graded sense. We observe that every simple superalgebra is prime.

Lemma 11. *Let A be a semiprime superalgebra. Then*

(i) (Brauer) *If I is a minimal right ideal of A, then there is an idempotent $e \in I_0$ such that $I = eA$. Moreover, for any homogeneous element $x \in I$ with $xI \neq 0$, there exists an idempotent $e = e^2 \in I$ such that $I = eA$ and $ex = xe = x$.*

(ii) *If e is a nonzero idempotent of A_0 and $eA = I$ is a minimal right ideal of A, then eAe is a division superalgebra, which is isomorphic to the centralizer superalgebra* $\operatorname{End}_A(I)$.

(iii) *If e is a nonzero idempotent of A_0 such that eAe is a division superalgebra, then eA is a minimal right ideal of A.*

(iv) *If a is a homogeneous element of A such that aA is a minimal right ideal of A, then Aa is a minimal left ideal of A.*

Proof. For (i), note that $I^2 \neq 0$ by semiprimeness, so $I^2 = I$ by minimality and there is a nonzero homogeneous element $x \in I_0 \cup I_1$ such that $xI \neq 0$. Again, since I is minimal, $I = xI$, and hence there is an element $e \in I_0$ such that $x = xe$. Take $J = \{r \in I\colon xr = 0\}$. Then J is a right ideal of A strictly contained in I, so $J = 0$. Since $e^2 - e \in J$, we conclude that e is a nonzero idempotent of I_0, and since $0 \neq eI \subseteq I$, the minimality of I forces $I = eI$, as desired. In particular $ey = y$ for any $y \in I$, so the last assertion follows.

For (ii) notice that if x is a homogeneous element of A such that $exe \neq 0$, then $0 \neq exeA \subseteq eA$ so $exeA = eA$ by minimality. Therefore there exists a homogeneous element $y \in A$ such that $exey = e$, so $(exe)(eye) = e$, which is the unity of the superalgebra eAe. Therefore, any nonzero homogeneous element of eAe has a right inverse. This is enough to ensure that eAe is a division superalgebra. Besides, the linear map $eAe \to \operatorname{End}_A(I)$ given by $exe \mapsto \rho_{exe}\colon I \to I$, such that $\rho_{exe}(z) = exez$ for any $z \in I$ is easily shown to be an isomorphism. Note that this is valid even if A is not semiprime.

Now assume that $0 \neq e = e^2 \in A_0$ satisfies that eAe is a division superalgebra, and let I be a nonzero right ideal contained in eA. Let x be a nonzero homogeneous element of I, so $x = ex$. If $exAe$ were 0, then $(AexA)^2$ would be 0 too, contradicting the semiprimeness of A. Therefore there is a homogeneous element $y \in A$ such that $exye \neq 0$, and since eAe is a division superalgebra, there is another homogeneous element z such that $xyeze = (exye)(eze) = e$. In particular, $e \in xA$ and $eA \subseteq xA \subseteq I$, so $I = eA$. This shows that eA is a minimal right ideal.

Finally, assume that a is a homogeneous element such that aA is a minimal right ideal. As in (i), let e be an idempotent such that $ae = ea = a$ and $aA = eA$. Because of (ii), eAe is a division superalgebra, and by symmetry, item (iii) shows that Ae is a minimal left ideal. But

$Aa \neq 0$ by semiprimeness, and $Aa = (Ae)a$ is a homomorphic image of the irreducible left module Ae, so it is irreducible too. That is, Aa is a minimal left ideal. □

The idempotents e such that eAe is a division superalgebra will be called *primitive idempotents*. Given a superinvolution $*$ in an associative superalgebra, $H(A,*)$ and $S(A,*)$ will denote, respectively, the set of fixed elements by $*$ and the set of elements $x \in A$ such that $x^* = -x$.

Theorem 12. *Let A be a prime superalgebra with minimal right ideals, and let $*$ be a superinvolution of A. Then one and only one of the following situations occurs*:

(i) *There exists a primitive idempotent such that $e^* = e$.*
(ii) $A_1 = 0$ *and there exists a primitive idempotent such that $eH(A,*)e^* = 0$. In this case eAe is a field and there are elements $u \in eAe^*$ and $v \in e^*Ae$ such that $u, v \in S(A,*)$, $uv = e$ and $vu = e^*$.*
(iii) *There exists a primitive idempotent such that $eA_0e^* = 0$. In this case, this idempotent e can be taken satisfying that there are elements $u \in eA_1e^*$ with $u^* = u$ and $v \in e^*A_1e$ with $v^* = -v$, such that $uv = e$ and $vu = e^*$.*

Proof. Assume that there is a minimal right ideal I and a homogeneous element $a \in I$ such that $aa^*I \neq 0$. Then take $x = aa^*$ $(\in I)$ and note that $x^* = (-1)^x x$. By Lemma 11 there is a primitive idempotent $e \in I$ with $I = eA$ and $xe = ex = x$. Therefore, $xee^* = xe^* = (-1)^x x^*e^* = (-1)^x (ex)^* = (-1)^x x^* = x$. Then, as in the proof of Lemma 11, $f = ee^*$ is an idempotent, with $f^* = f$ and $I = fA$ and case (i) appears.

Otherwise, for any minimal right ideal I of A, and any homogeneous element $a \in I$, $aa^*I = 0$ holds. Take a minimal right ideal I of A and assume that there exists a homogeneous element $a \in I$ such that $aa^* \neq 0$. By minimality, $I = aa^*A = aA$, and $I^*I = Aaa^*aa^*A \subseteq A(aa^*I) = 0$. By Lemma 11(iv), Aa is a minimal left ideal, and hence $a^*A = (Aa)^*$ is a minimal right ideal. Take $J = a^*A$. If there were a homogeneous element x in J with $xx^* \neq 0$, as before we would have $0 = J^*J = Aaa^*A$, but this is impossible since A is semiprime.

Hence, either the situation in (i) holds or there is a minimal right ideal I of A such that

$$xx^* = 0 \quad \text{for any homogeneous element } x \in I. \tag{13}$$

(Notice that up to now the arguments are valid assuming only that A is semiprime.)

Let I be such a minimal right ideal. By Lemma 11, $I = eA$ for a primitive idempotent e. Hence, for any homogeneous element $x \in A$, $exx^*e^* = (ex)(ex)^* = 0$. In particular, for any $x \in A_0$, $e(e+x)(e+x)^*e^* = 0 = ee^* = exx^*e^*$, so $e(x+x^*)e^* = 0$ (that is, $eH(A_0,*)e^* = 0$), or

$$(ex_0e^*)^* = -ex_0e^* \tag{14}$$

for any $x_0 \in A_0$. (Note that if $A_1 = 0$, this condition is equivalent to the condition in (13).)

Assume first that $eA_0e^* \neq 0$, and take $z \in A_0$ such that $eze^* \neq 0$. By primeness, $eze^*Ae \neq 0$, and since eAe is a division superalgebra, we can obtain easily another element $t \in A_0$ such that $eze^*te = e$. Take $u = eze^*$ and $v = e^*te$, so $uv = e$. Besides, $u^* = -u$ holds by (14).

Now, $v^2 = e^*t(ee^*)te = 0$ (by (13)), $u^2 = -uu^* = 0$, as $u \in I_0$, $e^*e = (uv)^*uv = v^*u^*uv = -v^*u^2v = 0$, and $v = e^*v = (uv)^*v = v^*u^*v = -v^*uv = -v^*e = -v^*$, so $e^* = v^*u^* =$

$(-v)(-u) = vu$. Let us denote by Δ the division superalgebra eAe. Consider the linear map $\Delta \to \Delta$: $d \mapsto \bar{d} = ud^*v$. Note that for any homogeneous $d, d_1, d_2 \in \Delta$:

$$\begin{aligned}\bar{\bar{d}} &= u(ud^*v)^*v = uv^*du^*v = uvduv = ede = d,\\ \overline{d_1d_2} &= u(d_1d_2)^*v = (-1)^{d_1d_2}ud_2^*d_1^*v = (-1)^{d_1d_2}ud_2^*e^*d_1^*v\\ &= (-1)^{d_1d_2}ud_2^*vud_1^*v = (-1)^{d_1d_2}\bar{d}_2\bar{d}_1.\end{aligned}$$

Therefore, this map is a superinvolution. But for any $d \in \Delta_0$,

$$\bar{d} = ud^*v = eud^*e^*v = -(eud^*e^*)^*v = -edu^*e^*v = eduv = ede = d,$$

where we have used (14), together with the fact that $u^* = -u$ and $uv = e$. Therefore the restriction of the superinvolution $d \mapsto \bar{d}$ is the identity, and since this is an ordinary involution of the division algebra Δ_0, we conclude that Δ_0 is a field. Besides, for any $d \in \Delta_1$ with $\bar{d} = \pm d$ (that is $d \in H(\Delta, -)_1 \cup S(\Delta, -)_1$), $d^2 \in \Delta_0$, so $d^2 = \overline{d^2}$. Hence

$$d^2 = \overline{d^2} = (-1)^{dd}\bar{d}^2 = -d^2.$$

Thus $d^2 = 0$, and since Δ is a division superalgebra, $d = 0$. Hence $\Delta_1 = H(\Delta, -)_1 \oplus S(\Delta, -)_1 = 0$, and $\Delta = \Delta_0$ is a field. But for any $x, y \in A_1$, $e(x+y)(x+y)^*e^* = 0 = exx^*e^* = eyy^*e^*$ by (13), so $exy^*e^* = -eyx^*e^*$. On the other hand, (14) shows that $(exy^*e^*)^* = -exy^*e^*$, that is, $-eyx^*e^* = -exy^*e^*$. We conclude that $exy^*e^* = 0$ for any $x, y \in A_1$. Therefore, $eA_1A_1e^* = 0$, so $eA_1A_1v = eA_1A_1e^*v = 0$. But $eA_1A_0v \subseteq eA_1e = \Delta_1 = 0$. Hence $eA_1Av = 0$, which implies, since A is prime, that $eA_1 = 0$. Now, $eAA_1 = eA_0A_1 + (eA_1)A_1 \subseteq eA_1 + (eA_1)A_1 = 0$, and $A_1 = 0$ by primeness. We are in case (ii) of the theorem, since $eH(A,*)e^*$ is 0 because of (14).

Finally, assume that the minimal right ideal $I = eA$ satisfies (13), but $eA_0e^* = 0$. Again, let Δ be the division superalgebra eAe. Since $eAe^* = eA_1e^* \neq 0$ and $\Delta_1eA_1e^* \subseteq eA_0e^* = 0$, and the nonzero elements of Δ_1 are invertible, it follows that $\Delta_1 = 0$ in this case.

If there exists an odd element $x \in A_1$ such that $e(x + x^*)e^* \neq 0$, then there is an element $z = x + x^* = z^* \in A_1$ such that $eze^* \neq 0$. As before we find an element $t \in A_1$ such that $eze^*te = e$ and take $u = eze^*$ and $v = e^*te$. Then $u^* = ez^*e^* = u$, $uu^* = 0 = ee^*$ because of (13), so $u^2 = 0$. Also, $v = e^*v = (uv)^*v = -(v^*u^*)v = -v^*uv = -v^*e = -v$, and $e^* = (uv)^* = -v^*u^* = vu$, thus obtaining the situation in item (iii) of the theorem.

Otherwise, for any $x \in A_1$ $e(x + x^*)e^* = 0$, or $(exe^*)^* = -(exe^*)$. As before there are $z, t \in A_1$ such that $eze^*te = e$. Take $u = eze^*$ and $v = e^*te$, so that $e = uv$, but now $u^* = -u$ and $v = e^*v = (uv)^*v = -v^*u^*v = v^*uv = v^*e = v^*$, and $e^* = (uv)^* = -v^*u^* = vu$. Consider in this case the primitive idempotent $f = e^*$. Then $fA_0f^* = vuA_0uv = ve(uA_0u)e^*v \subseteq v(eA_0e^*)v = 0$, so in particular the condition in (13) holds trivially for the minimal right ideal fA, and $f(v + v^*)f^* = 2fvf^* = 2v \neq 0$. Therefore, it is enough to change e to f to obtain the situation in item (iii).

To finish the proof of the theorem, it must be checked that only one of the three possible situations occur. It is clear that (ii) and (iii) are mutually exclusive, since $A_1 = 0$ in (ii) but not in (iii). Also, if e is a primitive idempotent as in (i), $I = eA$, and f is a primitive idempotent such that the minimal right ideal $J = fA$ satisfies the condition in (13), then by primeness $IJ \neq 0$, so by minimality $I = IJ$. Thus, there is a homogeneous element $x \in I$ such that $0 \neq xJ$ and

$I = xJ$ by minimality. Then $e = xz$ for some homogeneous element $z \in J$ of the same parity as x. But $e = e^2 = ee^* = (xz)(xz)^* = \pm xzz^*x^*$, which is 0 by (13), a contradiction. Therefore, case (i) is not compatible with cases (ii) or (iii). $\square$

Let $(\Delta, -)$ be a division superalgebra endowed with a superinvolution, let V be a left module over Δ and let $h : V \times V \to \Delta$ be an ϵ-hermitian ($\epsilon = \pm 1$) nondegenerate form of degree l ($l \in \{0, 1\}$). That is, h is biadditive, $h(V_i, V_j) \subseteq \Delta_{i+j+l}$ for any $i, j \in \{0, 1\}$ and

$$h(dx, y) = dh(x, y), \qquad h(y, x) = \epsilon(-1)^{xy}\overline{h(x, y)}$$

for any homogeneous elements $x, y \in V_0 \cup V_1$ and $d \in \Delta$. This implies that $h(x, dy) = (-1)^{dy} h(x, y)\bar{d}$ for any homogeneous $x, y \in V$ and $d \in \Delta$.

The case of $\Delta_1 = 0$, Δ_0 a field, $-$ the identity and $\epsilon = -1$ corresponds to the alternating bilinear forms over a field. If $\epsilon = 1$, h will just be said to be an hermitian form.

Consider the superalgebra with superinvolution $\mathcal{L}(V)$ with

$$\begin{aligned}\mathcal{L}(V)_i = \bigl\{ f \in \operatorname{End}_\Delta(V)_i \colon\ &\exists f^* \in \operatorname{End}_\Delta(V) \text{ such that} \\ &h(xf, y) = (-1)^{fy} h(x, yf^*)\ \forall x \in V,\ \forall y \in V_0 \cup V_1 \bigr\}.\end{aligned}$$

Note that f^* is unique by the nondegeneracy of f, which gives the superinvolution in $\mathcal{L}(V)$, and that the action of the elements of $\operatorname{End}_\Delta(V)$ is written on the right.

For any homogeneous elements $v, w \in V$, consider the Δ-linear map $h_{v,w}$ given by:

$$a \mapsto ah_{v,w} = h(a, v)w.$$

For any homogeneous $x, y, v, w \in V$,

$$\begin{aligned} h(xh_{v,w}, y) &= h(x, v)h(w, y) \\ &= (-1)^{v(w+y)} h\bigl(x, \overline{h(w, y)}v\bigr) \\ &= \epsilon(-1)^{v(w+y)+wy} h\bigl(x, h(y, w)v\bigr) \\ &= \epsilon(-1)^{v(w+y)+wy} h(x, yh_{w,v}), \end{aligned}$$

so that

$$(h_{v,w})^* = \epsilon(-1)^{vw} h_{w,v} \tag{15}$$

for any homogeneous $v, w \in V$. Also, for any homogeneous $f \in \mathcal{L}(V)$:

$$h_{v,w} f = h_{v,wf}, \qquad f h_{v,w} = (-1)^{fv} h_{vf^*,w}. \tag{16}$$

Therefore, the span $h_{V,V}$ of the $h_{v,w}$'s is an ideal of $\mathcal{L}(V)$ closed under the superinvolution, and it acts irreducibly on V. Besides, for any homogeneous element ψ in the centralizer $\operatorname{End}_{h_{V,V}}(V)$ (action on the left), and any homogeneous $x, v, w \in V$, $\psi(xh_{v,w}) = \psi(x)h_{v,w}$, so that, if $h(x, v) = 0$, then also $h(\psi x, v) = 0$. Since h is nondegenerate, there is a homogeneous element $d_x \in \Delta$ such that $\psi(x) = d_x x$. If now we take v with $h(x, v) = 1$, then we obtain that

$\psi(w) = d_x w$ for any homogeneous w. That is, ψ is the left multiplication by d_x. This shows that the centralizer of the action of $h_{V,V}$ on V is Δ.

Lemma 17. *Let A be an associative superalgebra and let V be an irreducible right A-module with centralizer $\Delta = \mathrm{End}_A(V)$ (action of Δ on the left), which is a division superalgebra by Schur's Lemma. Then for any homogeneous Δ-linearly independent elements $v_1, \ldots, v_n$ there is a homogeneous element $a \in A$ such that $v_1 a \neq 0$, $v_2 a = \cdots = v_n a = 0$.*

Proof. This is proved by induction on n, the case $n = 1$ being trivial. Assuming the result proven for $n - 1$, let $J = \{a \in A\colon\ v_3 a = \cdots = v_n a = 0\}$ be the right annihilator of $v_3, \ldots, v_n$. By the induction hypothesis $v_2 J \neq 0 \neq v_1 J$, and by irreducibility $V = v_2 J = v_1 J$. If there is a homogeneous element $a \in J$ with $v_2 a = 0 \neq v_1 a$, we are done. Otherwise, the map $\psi : V = v_2 J \to V = v_1 J$ such that $\psi(v_2 a) = v_1 a$ for any $a \in J$ is well defined and belongs to the centralizer $\mathrm{End}_A(V) = \Delta$, so that $\psi = d$ for some homogeneous element $d \in \Delta$. But then $(v_1 - d v_2) J = 0$, so by the induction hypothesis, $v_1 - d v_2 \in \Delta v_3 + \cdots + \Delta v_n$, a contradiction. $\square$

Corollary 18 *(Jacobson's density). Let A be an associative superalgebra and let V be an irreducible right A-module with centralizer $\Delta = \mathrm{End}_A(V)$ (action of Δ on the left). Then for any homogeneous Δ-linearly independent elements $v_1, \ldots, v_n$ and for any elements $w_1, \ldots, w_n$ there is an element $a \in A$ such that $v_i a = w_i$ for $i = 1, \ldots, n$.*

Corollary 19. *Let $(\Delta, -)$ be a division superalgebra endowed with a superinvolution, and let V be a left Δ-module endowed with a nondegenerate ϵ-hermitian form $h : V \times V \to \Delta$. Then $h_{V,V}$ is the only simple ideal of any subalgebra of $\mathcal{L}(V)$ containing it. In particular, any such subalgebra is prime.*

Proof. If f is a nonzero homogeneous element of $h_{V,V}$ and it is written as $f = \sum_{i=1}^n h_{v_i,w_i}$ (for homogeneous v_i, w_i, $i = 1, \ldots, n$), with minimal n, then $w_1, \ldots, w_n$ are linearly independent over Δ, so by Lemma 17 there is a homogeneous element $g \in h_{V,V}$ such that $\hat{w}_1 = w_1 g \neq 0$ and $w_2 g = \cdots = w_n g = 0$. Because of (16) $fg = h_{v_1,\hat{w}_1}$, and using again (16) it follows that $h_{V,V}$ is simple. Also, (16) shows that the ideal generated by any element of a subalgebra of $\mathcal{L}(V)$ containing $h_{V,V}$ intersects $h_{V,V}$ nontrivially. Hence the result. $\square$

The following result follows at once from (16):

Lemma 20. *With the same hypotheses as in Corollary* 19*, for any nonzero homogeneous element $v \in V$, $h_{v,V}$ is a minimal right ideal of any subalgebra of $\mathcal{L}(V)$ containing $h_{V,V}$.*

Note that, under the hypotheses of the lemma, a homogeneous element $w \in V$ can be taken with $h(w, v) = 1$. Then, the even element $e = h_{v,w} \in h_{V,V}$ is an idempotent with $e^* = \epsilon(-1)^{vw} h_{w,v}$ because of (16) and (15).

Theorem 21. *Let A be a prime superalgebra with minimal right ideals, and let $*$ be a superinvolution of A. Then one and only one of the following situations occurs*:

(i) *There exists a division superalgebra with a superinvolution $(\Delta, -)$, a left Δ-module V with $V_0 \neq 0$ endowed with a nondegenerate hermitian even superform $h : V \times V \to \Delta$,* *and a faithful representation $\rho : A \to \mathrm{End}_\Delta(V)$ such that $\rho(A)$ is a subalgebra of $\mathcal{L}(V)$, containing $h_{V,V}$, closed under the superinvolution of $\mathcal{L}(V)$, and such that $\rho(a^*) = \rho(a)^*$ for any $a \in A$.*

(ii) *$A_1 = 0$, and there is a field F and a vector space V over F endowed with a nondegenerate alternating bilinear form $h : V \times V \to F$ and a faithful representation $\rho : A \to \mathrm{End}_F(V)$ such that $\rho(A)$ is a subalgebra of $\mathcal{L}(V)$, containing $h_{V,V}$, closed under the superinvolution of $\mathcal{L}(V)$, and such that $\rho(a^*) = \rho(a)^*$ for any $a \in A$.*

(iii) *There exists a division algebra with an involution $(D, -)$, a left $\mathbb{Z}_2$-graded vector space V endowed with a nondegenerate hermitian odd form $h : V \times V \to D$, and a faithful representation $\rho : A \to \mathrm{End}_D(V)$ such that $\rho(A)$ is a subalgebra of $\mathcal{L}(V)$, containing $h_{V,V}$, closed under the superinvolution of $\mathcal{L}(V)$, and such that $\rho(a^*) = \rho(a)^*$ for any $a \in A$.*

Conversely, any such superalgebra is prime, contains minimal right ideals and it is endowed with a superinvolution.

Proof. Let A be a prime superalgebra with minimal right ideals, and let $*$ be a superinvolution of A. According to Theorem 12 three possible situations happen:

(i) There exists a primitive idempotent e with $e^* = e$. In this case, let $\Delta = eAe$, which is a division superalgebra with involution $-$ given by the restriction of $*$, let $V = eA$, let $h : V \times V \to \Delta$ given by $h(x, y) = xy^*$ for any $x, y \in V$, and let $\rho : A \to \mathrm{End}_\Delta(V)$ be the map given by $x \mapsto R_x$ (the right multiplication by x). It is clear that h is an even nondegenerate hermitian form. Note that for homogeneous $x = ea$, $y = eb$ and $z = ec$ in eA $(a, b, c \in A)$, $zh_{x,y} = h(z, x)y = zx^*y = za^*eb$, so $h_{V,V} = R_{AeA}$, which is obviously contained in R_A. Hence, all the conditions in (i) are satisfied.

(ii) $A_1 = 0$ and there exists a primitive idempotent such that $eH(A, *)e^* = 0$. In this case eAe is a field and there are elements $u \in eAe^*$ and $v \in e^*Ae$ such that $u, v \in S(A, *)$, $uv = e$ and $vu = e^*$ hold. Consider here the field $F = eAe$, let $V = eA$, and let $h : V \times V \to F$ given by $h(x, y) = xy^*v$. Since $eH(A, *)e^* = 0$, for any $x \in V$ $h(x, x) = (ex)(ex)^*v = exx^*e^*v \in eH(A, *)e^*v = 0$, so h is alternating, and nondegenerate by primeness. With $\rho(x) = R_x$ as before, the conditions in (ii) are satisfied.

(iii) There exists a primitive idempotent such that $eA_0e^* = 0$ and two elements $u \in eA_1e^*$ with $u^* = u$ and $v \in e^*A_1e$ with $v^* = -v$, such that $uv = e$ and $vu = e^*$. The proof of Theorem 12 shows that $eA_1e = 0$, so consider the division algebra $D = eAe = eA_0e$, the left vector space $V = eA$ and the map $h : V \times V \to D$ given by $h(x, y) = xy^*v$. Again, the proof of Theorem 12 shows that the map $- : D \to D$ given by $\bar{d} = ud^*v$ is an involution, and then h becomes an odd nondegenerate hermitian form, because for homogeneous elements $x, y \in V$,

$$\begin{aligned} h(y, x) &= yx^*v = eyx^*v = u(vyx^*)v \\ &= (-1)^{vy+vx+xy}u(xy^*v^*)v \\ &= -(-1)^{x+y+xy}u(xy^*v)v \quad \text{(as } v^* = -v \text{ and } v \text{ is odd)} \\ &= \overline{h(x, y)} \end{aligned}$$

as $h(x, y) = 0 = h(y, x)$ unless x and y have different parity. The conditions in (iii) are satisfied here.

Conversely, identifying A with $\rho(A)$ in all the cases, A is a prime superalgebra with superinvolution because of Corollary 19. Now for any nonzero homogeneous element $v \in V$, $I = h_{v,V}$ is a minimal right ideal of A (Lemma 20) generated by the primitive idempotent $e = h_{v,w}$, where w is a homogeneous element with $h(v,w) = 1$, which satisfies $e^* = (-1)^{vw} h_{w,v}$ (see the paragraph after Lemma 20).

Under the conditions of item (i), there is a nonzero element $v \in V_0$ such that $h(v,v) \neq 0$, because otherwise, for any $v, w \in V_0$, $h(v,w) = -h(w,v) = -\overline{h(v,w)}$, so the restriction of the superinvolution $-$ to Δ_0 would be minus the identity, which is impossible. Now, if $v \in V_0$ and $h(v,v) = d \neq 0$, then $h(w,v) = 1$ with $w = d^{-1}v$. Besides $\bar{d} = \overline{h(v,v)} = h(v,v) = d$, so with $e = h_{v,w}$, $e^* = h^*_{v,w} = h_{w,v} : x \mapsto h(x, d^{-1}v)v = h(x,v)d^{-1}v = xh_{v,w}$, and $e^* = e$. Thus A satisfies the hypotheses of Theorem 12(i).

Under the conditions of item (ii), let $e = h_{v,w}$ with $h(v,w) = 1$, so $e^* = h_{w,v}$. Now, for any $f \in H(A,*)$,

$$efe^* = h_{v,w} f h_{w,v} = h_{v,wf} h_{w,v} = h_{v,wfh_{w,v}} = h_{v,h(wf,w)v} = 0,$$

as $h(wf,w) = h(w,wf^*) = h(w,wf) = -h(wf,w)$, since h is alternating and $f \in H(A,*)$. Thus A satisfies the hypotheses of Theorem 12(ii).

Finally, under the conditions of item (iii), take $v \in V_0$ and $w \in V_1$ such that $h(v,w) = 1$, and take the primitive idempotent $e = h_{v,w}$. Then, for any $f \in \mathcal{L}(V)_0$, $efe^* = h_{v,h(wf,w)v} = 0$, as $h(w,wf) \in h(V_1,V_1) = 0$. Hence $eAe^* = 0$ and thus A satisfies the hypotheses of Theorem 12(iii).

Since the three possibilities in Theorem 12 are mutually exclusive, the same is valid here. □

Remark 22. If the condition of $V_0 \neq 0$ is omitted in item (i) of Theorem 21, then item (i) would include the situation of item (ii), as any alternating bilinear form on a vector space V over a field is an hermitian even form on the vector superspace $V = V_1$.

Because of Jacobson's Density (Corollary 18), if A is a finite dimensional associative superalgebra and V is an irreducible right module for A, then V is finite dimensional and A is isomorphic to $\mathrm{End}_\Delta(V)$, where $\Delta = \mathrm{End}_A(V)$ is a finite dimensional division superalgebra. Hence:

Corollary 23. *Let A be a finite dimensional simple superalgebra with $A_1 \neq 0$, and let $*$ be a superinvolution of A. Then one and only one of the following situations occurs*:

(i) *There exists a finite dimensional division superalgebra with a superinvolution $(\Delta, -)$, a finite dimensional left Δ-module V with $V_0 \neq 0$ endowed with a nondegenerate hermitian even superform $h : V \times V \to \Delta$, and an isomorphism of superalgebras with superinvolution $\rho : A \to \mathrm{End}_\Delta(V)$ (the superinvolution on $\mathrm{End}_\Delta(V)$ is the adjoint relative to the superform).*
(ii) *There exists a finite dimensional division algebra with an involution $(D, -)$, a finite dimensional left $\mathbb{Z}_2$-graded vector space V endowed with a nondegenerate hermitian odd form $h : V \times V \to D$, and an isomorphism of superalgebras with superinvolution $\rho : A \to \mathrm{End}_D(V)$.*

Conversely, any such superalgebra is simple and it is endowed with a superinvolution.

Notice that if A is a CSS and it is isomorphic to $\mathrm{End}_\Delta(V)$ for some finite dimensional division superalgebra Δ and a left vector space V over Δ, then the classes of A and Δ in the Brauer–Wall group coincide.

3. Superinvolutions of the first kind

In these sections all superantiautomorphisms and all superinvolutions are F-linear.

3.1. The odd case

First of all, we consider the quadratic graded algebras.

Lemma 24. *A quadratic graded algebra over a field F has a superantiautomorphism if and only if -1 is a square in F.*

Proof. Let $K := F \oplus Fu$, $u^2 = a \in F^\times$, $\delta u = 1$. We suppose that there is an element $s \in F$ with $s^2 = -1$. For $x := \alpha 1 + \beta u \in K$ $(\alpha, \beta \in F)$ we define $\sigma(x) = \alpha 1 + \beta s u$. The induced map σ is a superantiautomorphism of K.

On the other hand, if σ is a superantiautomorphism, then for all $\alpha, \beta \in F$ we have $\sigma(\alpha 1 + \beta u) = \alpha 1 + \beta \lambda u$. The relation $\sigma(u^2) = -\sigma(u)\sigma(u)$ implies that $\lambda^2 = -1$. $\square$

Lemma 25. *Quadratic graded algebras do not posses superinvolutions.*

Proof. We take a quadratic algebra $K := F \oplus Fu$, $u^2 = a \in F^\times$, $\delta u = 1$. If τ were a superinvolution, then $\tau(u) = \lambda u$ for some $\lambda \in F^\times$. The fact that $\tau^2(u) = u$ would imply that $\lambda = \pm 1$. Hence $a = \tau(a) = \tau(u^2) = -(\tau(u))^2 = -(\lambda u)^2 = -a$, a contradiction. $\square$

Remark 26. If a quadratic algebra has a superantiautomorphism φ, then $\varphi^2 = \nu$, the grading automorphism.

Now we can solve the case of algebras of odd type.

Theorem 27. *Let A be a CSS of odd type over a field F. The algebra A has a superantiautomorphism if and only if the following two conditions hold*:

1. -1 *is a square in F,*
2. *the algebra A_0 has an antiautomorphism.*

Proof. If A has a superantiautomorphism σ, then $\sigma|_{A_0}$ is an antiautomorphism of A. Moreover, the restriction $\sigma|_{Z(A)}$ is a superantiautomorphism on the quadratic graded algebra $Z(A)$. Hence Lemma 24 implies that -1 is a square in F.

Conversely, if -1 is a square in F then, again by Proposition 24, there is a superantiautomorphism σ_1 on $Z(A)$. If σ_2 is an antiautomorphism of A_0, then $\sigma_1 \otimes \sigma_2$ is a superantiautomorphism of $A \simeq (A_0) \otimes Z(A)$. $\square$

Theorem 28. *Let A be a CSS of odd type. The algebra A does not possess superinvolutions of the first kind.*

Proof. If τ were a superinvolution of A, then $\tau|_{Z(A)}$ would be a superinvolution on the quadratic graded algebra $Z(A)$, a contradiction with Lemma 25. $\square$

3.2. The even case

3.2.1. The case $Z(A_0) \simeq F \times F$

Let A be a CSS of even type with $Z(A_0) \simeq F \times F$. Then $A \simeq M_{n+m}(F) \mathbin{\widehat{\otimes}} (D)$, for a division algebra D. If $D = F$, we are in the case of Proposition 7: if $A \sim M_{n+m}(F)$, n, m odd and $n \neq m$ there is a superantiautomorphism but no superinvolution; otherwise A has a superantiautomorphism if and only if it has a superinvolution.

It remains to consider the case where $D \neq F$.

Theorem 29. *Let $D \neq F$. Then the following assertions are equivalent*:

(i) *A has a superantiautomorphism*
(ii) *A has a superinvolution*
(iii) *D has an involution*

Proof. The superalgebra A has a superantiautomorphism if and only if $A \mathbin{\widehat{\otimes}} A \sim 1$ in $BW(F)$. Hence $(D) \mathbin{\widehat{\otimes}} M_{n+m}(F) \mathbin{\widehat{\otimes}} M_{n+m}(F) \mathbin{\widehat{\otimes}} (D) \sim 1$ in $BW(F)$, also $(D) \mathbin{\widehat{\otimes}} (D) \sim 1$ in $BW(F)$. But since the grading on D is trivial, this is equivalent with $D \otimes D \sim 1$ in $B(F)$. By the classical Albert's Theorem, this is equivalent with the fact that D possess an involution. Hence (i) and (iii) are equivalent.

To prove that (iii) implies (ii), suppose that σ is an involution on D. Let $V := D^{n+m}$. If $e_1, \dots, e_{n+m}$ is the canonical basis of D^{n+m}, we define a grading on V by setting $e_1, \dots, e_n \in V_0$ and $e_{n+1}, \dots, e_m \in V_1$. Clearly, $A \simeq \mathrm{End}_D(V)$. The fact that $D \neq F$ implies that $\{d \in D \mid \sigma(d) = -d\} \neq \{0\}$ (if σ were the identity on D, then the central algebra D would coincide with F). Let $d \in D$ with $\sigma(d) = -d \neq 0$. The map $h : V \times V \to D$, given by $h(e_i, e_i) = 1$ for all $i = 1, \dots, n$, $h(e_j, e_j) = d$ for all $j = n+1, \dots, n+m$, and $h(e_i, e_j) = 0$ for $i \neq j$, defines a superhermitian form and therefore there is an associated superinvolution (see Corollary 23). Therefore (iii) implies (ii).

The implication (ii) $\Rightarrow$ (i) is trivial. □

3.2.2. The case $Z(A_0)$ is a field

We study CDS in relation with superinvolutions. We begin with graded quaternion algebras.

Lemma 30. *Let $Q = \langle a, b\rangle$ be a CDS. Then it has no superinvolution* (*of the first kind*).

Proof. Suppose that A has a superinvolution $*$. Then there is a nonzero element $u \in Q_1 \cap H(Q, *)$ or $u \in Q_1 \cap S(Q, *)$, because the eigenvalues of the linear map $*$ are ± 1 (or also: pick a nonzero element x in Q_1; if $x \notin S(Q, *)$, then take $x + x^* \in H(Q, *)$). Let $u^2 = \lambda \in F^\times$. Then $\lambda = u^2 = u^* u^* = -(u^2)^* = -(\lambda)^* = -\lambda$, which implies that $\lambda = 0$. Hence $u^2 = 0$, a contradiction to the fact that Q is a division superalgebra. □

Lemma 31. *Let Δ be a CDS of even type, $\Delta_1 \neq 0$. Then Δ has no superinvolutions.*

Proof. We prove the lemma by induction on the degree (as ungraded algebra) of Δ. The case of degree 2 is proved in Lemma 30.

Now, suppose that the degree of Δ is greater than 2 and that $*$ is a superinvolution on Δ. Since the eigenvalues of the linear map $*$ are ± 1, there is a nonzero element $u \in \Delta_1 \cap H(\Delta, *)$

or $u \in \Delta_1 \cap S(\Delta, *)$. Let $u^2 = a$. Then $a^* = (u^2)^* = -(u^*)^2 = -a$. Hence $a \notin F^\times$. On the other hand, since $au = ua$, it follows that $a \notin K := Z(\Delta_0)$, because $Z(\Delta_0) = F1 \oplus Fz$ and $xz = -zx$ for all $x \in \Delta_1$. We define the map $\sigma : \Delta_0 \to \Delta_0$, $\sigma(x) = uxu^{-1}$. Clearly, $\sigma^2(x) = axa^{-1}$. Let $G := \{x \in \Delta_0 \mid \sigma(x) = x\}$. The algebra G is a division subalgebra of Δ_0. The field $L = Z(G)$ contains a and hence it is a proper extension of F. Now we define $\tilde{\Delta} := C_\Delta(L)$, the centralizer of L in Δ, a proper division superalgebra contained in Δ.

We consider the irreducible representation of superalgebras $\rho : \Delta \otimes L \to \mathrm{End}_F(\Delta)$, $d \otimes l \mapsto \rho(d \otimes l)$, with $v\rho(d \otimes l)(v) := lvd$ for all $v \in \Delta$. Applying Jacobson's density (see Corollary 18), we have

$$\Delta \otimes L \simeq \mathrm{End}({}_{\tilde{\Delta}}\Delta) \simeq M_r(\tilde{\Delta}).$$

Therefore $L = Z(\tilde{\Delta})$, where $\tilde{\Delta}$ is an even CDS over L with degree (as algebra over L) less than the degree of the F algebra Δ. The map $* \otimes 1$ is a superinvolution on the L-superalgebra $\Delta \otimes L$. Hence also $M_r(\tilde{\Delta})$ has a superinvolution. It follows that the L-superalgebra $\tilde{\Delta}$ has a superinvolution (see Corollary 23(i)), a contradiction with the induction hypothesis. $\square$

Theorem 32. *Let A be a CSS of even type and suppose that $Z(A_0)$ is a field. Then it possesses no superinvolution.*

Proof. We know that (see Remark 5)

$$A \simeq \big(M_n(F)\big) \mathbin{\widehat{\otimes}} \Delta,$$

where Δ is a CDS. We observe, again by Corollary 23 that A has a superinvolution if and only if Δ has a superinvolution. Then the theorem follows immediately from Lemma 31. $\square$

Remark 33. We observe that even if a CSS A of even type where $Z(A_0)$ is a field has no superinvolution, it may have order two in the Brauer–Wall group. For example, if -1 is a square in F, then every (even) superalgebra of the form

$$F\langle\sqrt{a_1}\,\rangle \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} F\langle\sqrt{a_{2r}}\,\rangle$$

has an order two in the Brauer–Wall group because of Lemma 24. In particular, if -1 is a square in F, every division graded quaternion algebra has order two in $BW(F)$ but has no superinvolution.

3.3. An example: Clifford algebras

Let (V, q) be a quadratic space over the field F and let $C(V, q)$ denote its Clifford algebra. The Clifford algebra is in a canonical way a superalgebra: it possess the grading inherited from the grading of the tensor algebra. Moreover, the Clifford algebra is always endowed with a canonical involution, namely the involution fixing all the elements of V. For more details about Clifford algebras, see for example [7]. Thanks to our theorems, it is very easy to establish the existence of superinvolutions on Clifford algebras. It depends only on the dimension of V (denoted by $\dim q$) and the center of the even part of $C(V, q)$, denoted by $Z(C_0)$.

Corollary 34. *Let (V, q) be a quadratic space over the field F. Then*:

(i) *If $\dim q$ is odd, then there exists no superinvolution.*
(ii) *If $\dim q$ is even and $Z(C_0)$ is a field, then there exists no superinvolution.*
(iii) *If $\dim q$ is even and $Z(C_0) \simeq F \times F$, then there exists a superinvolution.*

4. The graded Albert Theorem

In this section we prove a graded version of Albert's Theorem. Here we have a CSS with superantiautomorphism and want to construct a superantiautomorphism whose square is the grading automorphism. Again, all the superantiautomorphisms in this section will be assumed to be F-linear.

In the first place, we define an invariant which will play an important role in the proof of the graded Albert Theorem.

Lemma 35. *Let A be a CSS of even type and suppose that the class of A has order $\leqslant 2$ in $BW(F)$. Let η be a superantiautomorphism of A. Then there is an invertible even element a such that $\eta^2(x) = axa^{-1}$ for any $x \in A$. Moreover, $a\eta(a) = \eta(a)a \in F^\times$ and the quantity $a\eta(a)F^{\times 2} \in F^\times/F^{\times 2}$ does not depend on the choice of η.*

Proof. Since η^2 is a graded automorphism which fixes elementwise $Z(A_0)$, the graded Skolem–Noether Theorem implies the existence of a homogeneous element $a \in A_0$ such that $\eta^2(x) = axa^{-1}$ for any x. We remark that $a\eta(a) \in F^\times$: this follows directly from the relation $\eta(\eta^2(x)) = \eta^3(x) = \eta^2(\eta(x))$ valid for every $x \in A$. Since η^2 fixes F, it follows that $a\eta(a) = \eta(a)a \in F^\times$.

Now, let ξ be another superantiautomorphism of A. Then there is a homogeneous element $b \in A^\times$ such that for all $x \in A$ we have $\xi(x) = (-1)^{bx} b\eta(x)b^{-1}$. A computation shows that $\xi^2(x) = b\eta(b)^{-1}axa^{-1}\eta(b)b^{-1}$. Let $c := b\eta(b)^{-1}a$. Then $c\xi(c) = a\eta(a)$. This number is uniquely determined modulo squares because a is uniquely determined up to a nonzero scalar in F. $\square$

Remark 36. In the ungraded case, A is a central simple F-algebra with involution of the first kind if and only if for any antiautomorphism σ of A (with $\sigma^2 = \iota_a$) we have $\sigma(a)a \in F^{\times 2}$ (see [12, p. 306]).

Lemma 37. *Let Δ be a CDS of even type, $\Delta_1 \neq 0$, and assume that the class of Δ has order $\leqslant 2$ in $BW(F)$. Then Δ has a superantiautomorphism φ such that $\varphi^2 = \nu$, the grading automorphism.*

Proof. Let η be a superantiautomorphism of Δ and $\eta^2(x) = axa^{-1}$, $a \in \Delta_0$. Now, let $Z(\Delta_0) = F \oplus Fz$, $z^2 = f \in F^\times$ with $f \notin F^{\times 2}$ and $\nu(x) = z^{-1}xz$. Moreover, we have $uz = -zu$ for all $u \in \Delta_1$.

We may assume that $\eta|_{Z(\Delta_0)} = \mathrm{id}|_{Z(\Delta_0)}$ (we know that $\eta(z) = \pm z$: if $\eta(z) = -z$, we pick a nonzero element $u \in \Delta_1$ and define the superantiautomorphism η' by $\eta' := \iota_u \circ \eta$, then $\eta'(z) = z$).

The map $\eta|_{\Delta_0}$ is a $Z(\Delta_0)$-linear antiautomorphism, hence by the classical Albert's Theorem Δ_0 has an involution. From Remark 36 it follows that $a\eta(a) \in Z(\Delta_0)^{\times 2}$.

Let $\lambda \in Z(\Delta_0)^\times$ be such that $a\eta(a) = \lambda^2$.

We may assume that $a \in Z(\Delta_0)^\times$. In fact, if $a \notin Z(\Delta_0)^\times$, then $1 + \lambda^{-1}a \neq 0$. We define $\xi = \iota_{(1+\lambda^{-1}a)^{-1}} \circ \eta$. Clearly, $\xi|_{Z(\Delta_0)} = \mathrm{id}|_{Z(\Delta_0)}$ and

$$\begin{aligned}\xi^2 &= \iota_{(1+\lambda^{-1}a)^{-1}} \circ \eta \circ \iota_{(1+\lambda^{-1}a)^{-1}} \circ \eta \\ &= \iota_{(1+\lambda^{-1}a)^{-1}} \circ \iota_{\eta(1+\lambda^{-1}a)} \circ \eta^2 \\ &= \iota_{(1+\lambda^{-1}a)^{-1}\eta(1+\lambda^{-1}a)a} \\ &= \iota_{(1+\lambda^{-1}a)^{-1}(a+\lambda^{-1}\eta(a)a)} \\ &= \iota_{(1+\lambda^{-1}a)^{-1}(a+\lambda)} \\ &= \iota_\lambda\end{aligned}$$

with $\lambda \in Z(\Delta_0)^\times$, as desired.

Since $a \in Z(\Delta_0)^\times$ and $\eta|_{Z(\Delta_0)} = \mathrm{id}|_{Z(\Delta_0)}$, we have $a\eta(a) = a^2 \in F^\times$. Hence $a \in F^\times$ or $a \in F^\times z$. But if $a \in F^\times$, then $\eta^2 = \mathrm{id}$, so it would be a superinvolution, and this contradicts Lemma 31. It follows that $a \in F^\times z$. Therefore $\eta^2 = \iota_z = \nu$, as desired. □

Theorem 38 *(Graded Albert Theorem). Let A be a CSS which possesses a superantiautomorphism. Then A has also a superantiautomorphism φ such that $\varphi^2 = \nu$, the grading automorphism.*

Proof. If A is odd, the existence of a superantiautomorphism implies that the algebra A_0 has an involution σ and that $\sqrt{-1} \in F$. Since $\sqrt{-1} \in F$, then the quadratic graded algebra $Z(A)$ has a superantiautomorphism s with $s^2 = \nu$ (see Remark 26). Hence the odd superalgebra $A \simeq (A_0) \mathbin{\widehat{\otimes}} Z(A)$ has the superantiautomorphism given by $\varphi = \sigma \otimes s$ such that $\varphi^2 = \nu$.

We follow the description of even CSS given in Remark 5.

If A is even and $A \simeq M_{r+s}(F) \mathbin{\widehat{\otimes}} (D)$, then D has an involution σ and $M_{r+s}(F)$ has a superantiautomorphism ϵ with $\epsilon^2 = \nu$ (see Proposition 7). Hence the superantiautomorphism $\varphi = \epsilon \otimes \sigma$ has the property $\varphi^2 = \nu$.

Finally, if A is even and $A \simeq (M_n(F)) \mathbin{\widehat{\otimes}} \Delta$, we apply Lemma 37 to Δ to get a superantiautomorphism ϵ with $\epsilon^2 = \nu$. Tensoring ϵ with the transpose, we get $\varphi := t \otimes \epsilon$ with $\varphi^2 = \nu$. □

Corollary 39. *Let A be an even CSS which possesses a superantiautomorphism η with $\eta^2 = \iota_a$. Then $Z(A_0) \simeq F(\sqrt{\eta(a)a})$.*

Proof. By Lemma 35 and Theorem 38 and the proof of Lemma 37 we may assume that $\eta^2 = \nu = \iota_z$, with $Z(\Delta_0) = F \oplus Fz$, $z^2 = f \in F^\times$ with $f \notin F^{\times 2}$ and $\eta(z) = z$. Then $\eta(z)z = z^2 = f$, and the result follows. □

5. Superinvolutions of the second kind

The situation for superinvolutions of the second kind, in contrast with the case of superinvolutions of the first kind, is analogous to the ungraded case in the sense that we can define a corestriction and prove that a superinvolution of the second kind exists if and only if the corestriction is trivial (the graded Albert–Riehm Theorem). It is also interesting to observe that superantiautomorphisms whose square is the grading automorphism do not always exist if the corestriction is trivial.

We consider a CSS A over a separable quadratic field extension $K = F(\theta)$ with Galois group $\mathrm{Gal}(K/F) = \{1, j\}$ and $\theta^2 \in F \setminus F^2$, $\bar\theta := j(\theta) = -\theta$.

A K/F-*superantiautomorphism* ξ of A is a superantiautomorphism which is K/F-semilinear, i.e. $\xi(\lambda x) = \bar{\lambda}\xi(x)$ for all $\lambda \in K$, $x \in A$. Accordingly, a K/F-superantiautomorphism ξ is called K/F-*superinvolution* if $\xi^2(x) = x$ for all $x \in A$.

Example 40. Let $A := M_{n+m}(K)$, $K = F(\theta)$. Then the CSS A has always a superinvolution of the second kind, namely the superinvolution adjoint to the hermitian superform on K^{n+m} given by the diagonal matrix $\mathrm{diag}(1, \dots, 1, \theta, \dots, \theta)$.

Let $\bar{A}$ be the superalgebra which is identical with A^s as ring but with the action of K twisted by j. Let $T := A \mathbin{\widehat{\otimes}} \bar{A}$. We remark that T is an even CSS over K. We consider the map $\pi : T \to T$, defined in the following way: for all homogeneous elements $p \in A$ and $q \in \bar{A}$ we have $p \otimes q \mapsto (-1)^{pq} q \otimes p$. We now define

$$\mathrm{cor}_{K/F}(A) = \{x \in T \mid \pi(x) = x\}.$$

Since $T = \mathrm{cor}_{K/F}(A) + \theta\,\mathrm{cor}_{K/F}(A) \simeq \mathrm{cor}_{K/F}(A) \otimes_F K$, we conclude that $\mathrm{cor}_{K/F}(A)$ is an even CSS over F. Of course, if $A = A_0$, the corestriction $\mathrm{cor}_{K/F}(A)$ coincides with the usual ungraded corestriction (see [12, p. 308]).

Example 41. The corestriction of the quadratic graded algebra $A := K\langle\sqrt{\mu}\rangle$ (recall that $K\langle\sqrt{\mu}\rangle = K \oplus Ku$ with the relation $u^2 = \mu$ and the grading $\delta u = 1$) is the linear hull of the set $\{1 \mathbin{\widehat{\otimes}} 1, \theta u \mathbin{\widehat{\otimes}} u, u \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} u, \theta u \mathbin{\widehat{\otimes}} 1 - 1 \mathbin{\widehat{\otimes}} \theta u\}$. Since $\mathrm{cor}_{K/F}(A) \otimes_F K \simeq A \mathbin{\widehat{\otimes}}_K \bar{A}$, which is a graded quaternion algebra over K, then $\mathrm{cor}_{K/F}(A)$ is a graded quaternion algebra over F. Here, $\mathrm{cor}_{K/F}(A)_0$ is the linear hull of $\{1 \mathbin{\widehat{\otimes}} 1, \theta u \mathbin{\widehat{\otimes}} u\}$. Now $(\theta u \mathbin{\widehat{\otimes}} u)^2 = -(\theta u)^2 \mathbin{\widehat{\otimes}} u^2 = -\theta^2\mu \mathbin{\widehat{\otimes}} \mu = -\theta^2\mu\bar{\mu}(1 \mathbin{\widehat{\otimes}} 1) = \mathrm{N}_{K/F}(\theta\mu)(1 \mathbin{\widehat{\otimes}} 1)$. Hence $\mathrm{cor}_{K/F}(A)_0 \simeq F[\sqrt{\mathrm{N}_{K/F}(\theta\mu)}\,]$.

The next step is the definition of a (right) $\mathrm{cor}_{K/F}(A)$-module structure on A when A possesses a K/F-superantiautomorphism.

In particular, if ξ is a K/F-superantiautomorphism, then we define a right action of T on A:

$$\Xi : T \to \mathrm{End}_K(A),$$

given by

$$x \cdot (p \otimes q) = x(p \otimes q)^{\Xi} := (-1)^{px}\xi(p)xq$$

for all homogeneous $p \in A$, $q \in \bar{A}$ and $x \in A$. The map Ξ is an isomorphism (because T is a CSS and by dimension count).

We will often use the equivalences A has a K/F-superantiautomorphism $\Leftrightarrow$ the K-CSSs $\bar{A}$ and A^s are isomorphic $\Leftrightarrow$ $T \simeq \mathrm{End}_K(A) \sim 1$ in $BW(F)$.

Now A is an irreducible (right) T-module. Since

$$\mathrm{cor}_{K/F}(A) \hookrightarrow T \simeq \mathrm{End}_K(A) \hookrightarrow \mathrm{End}_F(A),$$

we have a (right) $\mathrm{cor}_{K/F}(A)$-module structure on A.

Lemma 42. *Let A be a CSS over K with a K/F-superantiautomorphism ξ. Let $C := \mathrm{End}_{\mathrm{cor}_{K/F}(A)}(A)$. Then $\mathrm{cor}_{K/F}(A) \sim C$ in $BW(F)$ and $\dim_F C = 4$, $K \subseteq C_0$. Moreover, $\mathrm{cor}_{K/F}(A) \sim 1$ if and only if A is not irreducible as $\mathrm{cor}_{K/F}(A)$-module.*

Proof. The fact that $K \subseteq C_0$ follows immediately from $\mathrm{cor}_{K/F}(A) \subseteq T \simeq \mathrm{End}_K(A)$.

First, we suppose that there exists a nontrivial proper irreducible $\mathrm{cor}_{K/F}(A)$-submodule of A. Let $V \neq 0$ be an irreducible $\mathrm{cor}_{K/F}(A)$-submodule of A with $V \neq A$. Then $KV := V \oplus \theta V$ is a $K \otimes_F \mathrm{cor}_{K/F}(A) \simeq T$-submodule, hence $KV = A$. By dimension count, $\mathrm{cor}_{K/F}(A) \simeq \mathrm{End}_F(V)$. Therefore $A \simeq V \oplus V$ (as $\mathrm{cor}_{K/F}(A)$-module). Hence we conclude that $\mathrm{End}_{\mathrm{cor}_{K/F}(A)}(A)$ is a trivially graded split quaternion algebra. This implies that $\mathrm{cor}_{K/F}(A) \sim 1$ in $BW(F)$. Now, let A be an irreducible $\mathrm{cor}_{K/F}(A)$-module. Then, by Schur's Lemma, C is a CDS and by Jacobson density we have that $\mathrm{cor}_{K/F}(A) = \mathrm{End}_C(A)$. Since $(\dim_K A)^2 = \dim_F \mathrm{cor}_{K/F}(A) = \dim_F \mathrm{End}_C(A) = (\dim_C A)^2 \dim_F C$, we have that $\dim_F C = (\frac{\dim_K A}{\dim_C A})^2 = (\dim_K C)^2$. This equation and the observation that $\dim_F C = (\dim_K C)(\dim_F K) = 2\dim_K C$ imply that $\dim_K C = 2$ and hence $\dim_F C = 4$. We remark that if $C_1 \neq 0$, then $K = C_0$. □

Lemma 43. *Let A be a CSS over K with a K/F-superantiautomorphism ξ. Then one and only one of the following cases occurs*:

(i) *Either $\xi^2 = \iota_b$, with $b \in A_0^\times$. In this case $\mathrm{cor}_{K/F}(A) \sim Q$ in $BW(F)$, where $Q = Q_0 = K \oplus Ku$ is a quaternion algebra with $u^2 = \xi(b)b \in F^\times$ and $u\alpha = \bar{\alpha}u$ for any $\alpha \in K$,*
(ii) *or $\xi^2 = \iota_b$, with $b \in A_1^\times$. In this case $\mathrm{cor}_{K/F}(A) \sim H$ in $BW(F)$, with H a quaternion CDS and $H_0 = K$.*

Proof. The dichotomy is given by the observation that ξ^2 is a graded automorphism and the Skolem–Noether Theorem. Recall from Lemma 42 that A is a right module for $\mathrm{cor}_{K/F}(A)$, and thus the action of the centralizer $C := \mathrm{End}_{\mathrm{cor}_{K/F}(A)}(A)$ will be written on the left. Also, K is a subalgebra of C_0, and the action (by scalar multiplication) of any element $\alpha \in K$ will be denoted by l_α: $l_\alpha(a) = \alpha a$ for any $a \in A$. To prove the lemma it is sufficient to compute C (see Lemma 42). We already know that C is either a quaternion CDS or a quaternion algebra with trivial grading.

Let $\xi^2 = \iota_b$, with $b \in A_0^\times \sqcup A_1^\times$. Note first that for any homogeneous $x \in A$, $\xi^3(x) = \xi(\xi^2(x)) = \xi((-1)^{bx}bxb^{-1}) = (-1)^{bx}\xi(b)^{-1}\xi(x)\xi(b)$, but also $\xi^3(x) = \xi^2(\xi(x)) = (-1)^{bx}b\xi(x)b^{-1}$. Thus $\xi(b)b$ is an even element in $Z(A) = K$, and it is fixed by ξ, so $\xi(b)b \in F^\times$.

Consider the F-linear map $f : A \to A$ given by $f(x) = (-1)^{xb}\xi(x)b$ for any homogeneous $x \in A$. The map f is even or odd, according to b being even or odd. For any $\alpha \in K$ and $x \in A$,

$$f \circ l_\alpha(x) = f\bigl(\alpha(x)\bigr) = \bar{\alpha} f(x) = l_{\bar{\alpha}} \circ f(x),$$

and

$$f^2(x) = \xi\bigl(\xi(x)b\bigr)b = (-1)^{bx}\xi(b)\xi^2(x)b = \xi(b)bxb^{-1}b = \xi(b)bx,$$

so $f^2 = l_{\xi(b)b}$ and the algebra over F generated by K and f (a subalgebra of $\mathrm{End}_F(A)$) is $K \oplus Kf$, which is a quaternion (ungraded) algebra if b is even, or a quaternion superalgebra with even part K if b is odd. It remains to be shown that f lies in the centralizer C. For any homogeneous elements $x, p, q \in A$,

$$\begin{aligned}
&f\big(x \cdot \big(p \otimes q + (-1)^{pq} q \otimes p\big)\big)\\
&\quad = f\big((-1)^{px}\xi(p)xq + (-1)^{(p+x)q}\xi(q)xp\big)\\
&\quad = (-1)^{(p+x+q)b}\big((-1)^{px}\xi\big(\xi(p)xq\big)b + (-1)^{(p+x)q}\xi\big(\xi(q)xp\big)b\big)\\
&\quad = (-1)^{(p+x+q)b}\big((-1)^{(p+x)q}\xi(q)\xi(x)\xi^2(p)b + (-1)^{px}\xi(p)\xi(x)\xi^2(q)b\big)\\
&\quad = (-1)^{(p+x+q)b}\big((-1)^{(p+x)q}(-1)^{pb}\xi(q)\xi(x)bp + (-1)^{px}(-1)^{qb}\xi(p)\xi(x)bq\big)\\
&\quad = (-1)^{p(x+b)}\xi(p)\big((-1)^{xb}\xi(x)b\big)q + (-1)^{q(p+x+b)}\xi(q)\big((-1)^{xb}\xi(x)b\big)p\\
&\quad = f(x) \cdot \big(p \otimes q + (-1)^{pq} q \otimes p\big).
\end{aligned}$$

Since the elements $p \otimes q + (-1)^{pq} q \otimes p$ span $\mathrm{cor}_{K/F}(A)$, we conclude that f is in the centralizer C, as required. $\square$

Theorem 44 *(Graded Albert–Riehm Theorem). Let K/F be a quadratic field extension. Let A be a CSS over K. Then A has a K/F-superinvolution if and only if* $\mathrm{cor}_{K/F}(A) \sim 1$ *in $BW(F)$.*

Proof. First, we assume that ξ is a K/F-superinvolution on A. Clearly, $\xi^2 = \iota_1$. By Lemma 43, $\mathrm{cor}_{K/F}(A) \sim Q$, with $Q = K \oplus Ku$, $u^2 = \xi(1)1 = 1$, $Q = Q_0$. But Q is split, hence $\mathrm{cor}_{K/F}(A) \sim 1$ in $BW(F)$.

To prove the second implication, we suppose that $\mathrm{cor}_{K/F}(A) \sim 1$ in $BW(F)$. Then $K \otimes \mathrm{cor}_{K/F}(A) \simeq T \sim 1$ in $BW(K)$. Hence $\bar{A} \simeq A^s$.

We know (see Theorem 4 and Remark 5) that either A is even and there is a central division K-algebra D such that $A \simeq M_{n+m}(K) \mathbin{\widehat{\otimes}} (D)$ (case I) or there exists a K-CDS Δ (even if A is even, odd otherwise) such that $A \simeq (M_n(K)) \mathbin{\widehat{\otimes}} \Delta$ (case II).

We remark that the CSSs $M_{n+m}(K)$ and $(M_n(K))$ have a K/F-superinvolution. Hence, by the first implication, they have trivial corestriction.

We consider case I. If $A \simeq M_{n+m}(K) \mathbin{\widehat{\otimes}} (D)$, then (see [12, p. 310])

$$\mathrm{cor}_{K/F}(A) \simeq \mathrm{cor}_{K/F}\big(M_{n+m}(K)\big) \mathbin{\widehat{\otimes}} \mathrm{cor}_{K/F}\big((D)\big).$$

Hence in case I we have $\mathrm{cor}_{K/F}((D)) \sim 1$ in $BW(F)$ and since the grading of D is trivial the classical Albert–Riehm Theorem implies that D has an involution of the second kind.

To conclude the proof, we have to study case II. As in case one, we obtain that $\mathrm{cor}_{K/F}(\Delta) \sim 1$ in $BW(F)$. Let ξ be a K/F-superantiautomorphism of Δ (it exists because if $\mathrm{cor}_{K/F}(\Delta) \sim 1$ then $\bar{\Delta} \simeq \Delta^s$). Since $\mathrm{cor}_{K/F}(\Delta) \sim 1$, Lemma 43 forces $\xi^2 = \iota_b$, with $b \in A_0^\times$. Moreover, we may suppose that $\xi(b)b = 1$ (because the fact that $\mathrm{cor}_{K/F}(\Delta) \sim Q \sim 1$ implies that there is an element $\lambda \in K$ such that $\lambda\bar{\lambda} = \xi(b)b$ and we may change b with $\lambda^{-1}b$). If $b = -1$, we have finished the proof. If $b \neq -1$, then the map $\eta = \iota_{(1+b)^{-1}} \circ \xi$ is a superinvolution. $\square$

Example 45. We consider the quadratic graded algebra $A := K\langle\sqrt{\mu}\rangle$ (with $K\langle\sqrt{\mu}\rangle = K \oplus Ku$ with the relation $u^2 = \mu$ and the grading $\delta u = 1$). We have computed the corestriction in Example 41: we know that $\mathrm{cor}_{K/F}(A) \sim 1$ in $BW(F)$ if and only if $\mathrm{N}_{K/F}(\theta\mu)$ is a square in F.

This condition is equivalent with the existence of an odd element v such that $v^2 \in F^\times\theta$ (or equivalently, to the existence of an element $\alpha \in K^\times$ such that $\alpha^2\mu \in F^\times\theta$). In fact, if $\alpha^2\mu = \gamma\theta$, $\gamma \in F^\times$, then $\mathrm{N}_{K/F}(\theta\mu) = \mathrm{N}_{K/F}(\alpha^{-2}\gamma\theta^2) = \mathrm{N}_{K/F}(\alpha^{-1})^2(\gamma\theta^2)^2 \in F^{\times 2}$. On the other hand, if $\mathrm{N}_{K/F}(\theta\mu) = \gamma^2$, $\gamma \in F^\times$ we have $\mathrm{N}_{K/F}(\gamma^{-1}\theta\mu) = 1$ and by Hilbert 90 there is an element $\delta \in K^\times$ with $\gamma^{-1}\theta\mu = \delta\bar{\delta}^{-1}$. Hence $\bar{\delta}^2\mu = (\gamma\theta^{-2}\delta\bar{\delta})\theta \in F^\times\theta$.

Remark 46. The preceding result can also be proved directly.

Proposition 47. *The CSS $A = K\langle\sqrt{\mu}\rangle$ has a K/F-superinvolution if and only if there is an element $v \in A_1$ such that $v^2 \in F^\times\theta$.*

Proof. Let $*$ be a K/F-superinvolution and $K\langle\sqrt{\mu}\rangle = K \oplus Ku$. Since the eigenvalues of $*$ are ± 1, we may assume $u^* = \pm u$. Then $\mu^* = (u^2)^* = -u^*u^* = -\mu$. Hence $\mu \in F^\times\theta$. Conversely, if there is an element $v \in A_1$ such that $v^2 \in F^\times\theta$, then the map $* : x + yv \mapsto \bar{x} + \bar{y}v$ $(x, y \in K)$ defines a K/F-superinvolution of A. □

For CSSs of odd type there is a criterion for the existence of superinvolutions of the second kind which is easier than the general one in terms of corestriction.

Proposition 48. *Let A be a CSS of odd type over K. Then A has a K/F-superinvolution if and only if the following two conditions are satisfied*:

(i) *The CSS $Z(A)$ has a K/F-superinvolution.*
(ii) *The central simple algebra A_0 has an involution of the second kind.*

Proof. We give two different proofs: one using Theorem 44 and a direct proof. First, we give the proof using Theorem 44, which shows how the proposition is related to the general result.

If $Z(A)$ has a K/F-superinvolution and A_0 has an involution of the second kind, then $\mathrm{cor}_{K/F}(Z(A)) \sim 1$ and $\mathrm{cor}_{K/F}(A_0) \sim 1$. Hence

$$\mathrm{cor}_{K/F}(A) \simeq \mathrm{cor}_{K/F}\big(Z(A)\big) \mathbin{\widehat{\otimes}} \mathrm{cor}_{K/F}(A_0) \sim 1,$$

as desired. Conversely, suppose that $\mathrm{cor}_{K/F}(A) \sim 1$. We know (see Lemma 43 and Proposition 47) that $\mathrm{cor}_{K/F}(A_0) \sim Q$, a quaternion algebra with $Q = Q_0$, $K \subseteq Q$ and $\mathrm{cor}_{K/F}(Z(A)) \sim H$, a graded quaternion algebra. We have only two possibilities: either $Q \sim 1$ and $H \sim 1$ or Q is a division algebra and $H \simeq Q^s = Q^{op}$. This last case cannot occur, because $H_1 \neq 0 = Q_1^{op}$. Hence we conclude that $\mathrm{cor}_{K/F}(A) \sim 1$ if and only if $\mathrm{cor}_{K/F}(Z(A)) \sim 1$ and $\mathrm{cor}_{K/F}(A_0) \sim 1$.

Now we give an easier direct proof.

Clearly, if the $Z(A)$ has a K/F-superinvolution τ_1 and the central simple algebra A_0 has an involution τ_2 of the second kind, then $\tau_1 \otimes \tau_2$ is a K/F-superinvolution on $A = Z(A) \mathbin{\widehat{\otimes}} A_0$. On the other hand, if τ is a K/F-superinvolution on A, then $\tau|_{A_0}$ is an involution of the second kind on A_0 and $\tau|_{Z(A)}$ is a K/F-superinvolution on $Z(A)$. □

Example 49. The existence of a superinvolution does not imply the existence of a superantiautomorphism whose square is the grading automorphism. For example, consider the quadratic graded algebra $K\langle\sqrt{i}\rangle = K \oplus Ku$ with $K = \mathbb{Q}(i)$, $i^2 = -1$. Clearly, it has a superinvolution.

But suppose that φ is a graded superantiautomorphism with $\varphi^2 = \nu$. Then $\varphi^2(u) = -u$. On the other hand, there is a $\lambda \in K$ such that $\varphi(u) = \lambda u$. Hence $-u = \varphi^2(u) = \varphi(\lambda u) = \bar{\lambda}\lambda u$. But the equation $\bar{\lambda}\lambda = -1$ has no solution in K, a contradiction to the existence of φ.

Example 50. We give an example of even CSS with superinvolution but with no superantiautomorphism whose square is the grading automorphism. Let $F = \mathbb{Q}$, $K = F(i)$, $i^2 = -1$ and $A = K\langle\sqrt{i}\rangle \hat{\otimes} K\langle\sqrt{3i}\rangle$. The CSS A has a superinvolution (and hence $\mathrm{cor}_{K/F}(A) \sim 1$), because $K\langle\sqrt{i}\rangle$ and $K\langle\sqrt{3i}\rangle$ possess a superinvolution. Here $A_0 = K1 \oplus Kz$ with $z^2 = 3$. Now, suppose that φ is a graded superantiautomorphism with $\varphi^2 = \nu$. Then $\varphi^2 = \iota_z$ and $\varphi(z) \in A_0$ and $\varphi(z)^2 = \varphi(z^2) = 3$. Moreover, $\varphi(z) = \pm z$. We know (see Lemma 43) that $\mathrm{cor}_{K/F}(A) \sim K \oplus Kf$ with $f^2 = \varphi(z)z = \pm z^2 = \pm 3$. But $\pm 3 \notin \mathrm{N}_{K/F}(K)$, hence $K \oplus Kf$ is a division algebra, a contradiction with $\mathrm{cor}_{K/F}(A) \sim 1$.

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