BACKWARD REASONING AND EPISTEMIC ACTIONS IN STRATEGIC GAMES PROBLEMS

Marta Barbero[1,2] Inés M. Gómez-Chacón[2] and Ferdinando Arzarello[11]

Università di Torino (Italy), [2]Universidad Complutense de Madrid (Spain), igomezchacon@mat.ucm.es

This paper focuses on the epistemic and cognitive characterization of backward reasoning in strategy games resolution. It explores the use of AiC (Abstraction in Context) as a tool for the analysis of the epistemic actions involved in these processes. It is reported a first analysis developed by the research team in order to be used as protocol-guide in the analysis of a study carried out with PhD students in Mathematics Degree in a Spanish and an Italian University, who face problem solving games. The case study shows the process of discovery that a PhD student makes to formulate a general recursive formula. It is a key for understanding the interaction between the AiC model and the characteristics of backward reasoning. The analysis allows to combine the two models - backward reasoning and AiC - in a unified framework that allows to focus both short-term and long-term processes in students' activities.

Keywords: Teaching and learning of specific topics in university mathematics, Teaching and learning of logic, reasoning and proof, backward reasoning, epistemic actions, strategy games

INTRODUCTION

Backward reasoning has great potential in the study of mathematics since it can support students when engaged in tasks, where they are asked to pass from argumentations and inquiry to mathematical proofs. For deepening this issue we specifically developed some studies at the university level focussed on mathematical thinking, where learning the method of analysis is a critical issue (Antonini, 2011, Peckhaus, 2000).

In such studies, which analysed mathematics and engineering students involved in problem solving activities (Gómez-Chacón & Barbero, 2018 & 2019), it was noted that so-called regressive reasoning — as an emerging key process in the dialectics between inference processes — develops mainly in interrogative movements and is responsible for the generation of new ideas and elements in the solution process. This reasoning is used in its character of "ordering device": through it, the students manage to find elements necessary for the construction/definition of the objective. The backward reasoning, which is based on the return of reasoning to an informal context, helps to connect more intuitive aspects with the mathematical and computational context.

In standard mathematical problems, it is more difficult working backwards than forwards. So it is necessary to offer students a large class of problems to which the method of working backwards is appropriate, such as strategy games presented here. We also identified some factors in the cognitive and affect interplay, which would inevitably cause difficulties for students to construct and work backwards. These studies (Gómez-Chacón, 2017) showed how the epistemic emotions continually exert numerous so-called operator effects, both linear and nonlinear, on attentional activity and on the ability to perceive goal-path obstacles and to overcome them. Understanding is linked with the appraisal of their ability to influence (control dimension), with their ability to predict, and with mental flexibility (Gómez-Chacón, 2017; Gómez-Chacón & Barbero, 2019). The dectected taxonomy of obstacles suggests that the lecturer, as a mediator of knowledge, explicitly takes into account the nature of backward reasoning underlying the interplay between epistemological and cognitive models.

This paper focuses on the epistemic and cognitive characterization of backward reasoning in strategy games resolution. Strategy games allow for the natural development of backward reasoning. Players must make strategic choices to make their moves. These choices are triggered by typical implicit questions that players ask before making a new move: “What can I do in this situation? What is better to do?” To answer these typically strategic questions, they reflect both on the moves already made and on the possible moves to do and they activate the backward reasoning (Wickelgren, 1974).

We explore the use of AiC (Abstraction in Context) as a tool for the analysis of the epistemic actions involved in these processes of resolution. We try to understand how the process of abstraction evolves, analyzing the relationship established between the epistemic actions (categories) of the RBC model of Dreyfus and Kidron (2014), and it is based on the perspective of abstraction in AiC context (Dreyfus et al., 2001), as well as by the subcategories of analysis introduced in this investigation based on the specific characteristics of the regressive reasoning. Epistemic actions are understood as mental actions that develop during the abstraction process and explain the emergence of a new, more elaborate and complex construction. We report a first analysis developed by the research team in order to be used as protocol-guide in the analysis of a study carried out with 185 undergraduate students in Mathematics Degree. Further analysis can be found in Barbero, Gómez-Chacón & Arzarello (2020).

The structured as follows: first the theoretical frame underlying the analytical methodology of the study; second, the context of the study description and its particular goals; third, first results presentation, drawn from a case study micro-analysis, where the theoretical background is applied; finally, a discussion and some conclusions.

BACKWARD REASONING

In mathematics, progressive reasoning alone is not exhaustive to fulfil the tasks of solving problems. Great mathematicians like Pappus, Descartes, Leibniz, in their discussions about analysis and synthesis, emphasize this fact (Peckhaus, 2000). Backward reasoning is known by different denominations, each underlying some of its main features: regressive analysis, backward solution, method of analysis, etc. It is the practice that involves the making of a number of arguments from the bottom of the problem and proceeds through logical correspondences which allow to obtain something known or to be reached through other paths. This process includes different ways of proceeding in problem solving: Backward heuristics, Reductio ad Absurdum, Starting with the end of the problem, Assuming the problem solved (Beaney, 2018).

Pappus was the mathematician who has contributed substantially to the clarification and exemplification of the method. In the seventh book of his Collection he deals with the topic of Heuristics (methods to solve the problems). There he exemplifies the method of analysis as the method of synthesis, therefore making the development of this reasoning clearer. Pappus defines the method of analysis as follows: "In analysis, we start from what is required, we take it for granted; and we draw correspondence (ακολουθον) from it and correspondence from the correspondence, till we reach a point that we can use as a starting point in synthesis. That is to say, in analysis we assume what is sought as already found (what we have to prove as true)." (elaboration by Polya, 1965 and by Hintikka and Remes, 1974). Subsequently he points out: "This procedure we call analysis, or solution backward, or regressive reasoning." (Hintikka and Remes, 1974). And on the Method of Synthesis: "In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. This procedure we call synthesis, or constructive solution, or progressive reasoning." (Hintikka and Remes, 1974)

The two processes are closely related and there is no analysis method without the synthesis one. Solving a problem is therefore a combination of the two procedures. Peckhaus (2000) studies this analysis-synthesis scheme and affirms that "The analytical [is] […] the procedure which starts with the formulation of the problem and ends with the determination of the conditions for its solution. The synthetical represents the way from the conditions to the actual solution of the problem. […] This branch of the scheme is deeply connected with the complementary [one]." Not only analysis can't exist without synthesis but also "synthesis can't be isolated and presupposes analysis."

The concept of Backward Reasoning involves characteristics that allow us to identify its development throughout the resolution of a task. Philosophers and mathematicians from the ancient Greeks, through the authors from the 17th and 18th centuries to the 20th one have studied its characteristics. The main features are the following:

- *Direction vs cause-effect*. In Pappus' definition, the backward direction of reasoning is highlighted. This entails going from the end of the problem to its beginning. By applying the method, the premises of a certain idea are sought. In the 17th and 18th centuries, authors such as Arnauld and Nicole interpreted the method as a search for cause-effect relationships between ideas. By these, the connection between the notions in background and the problem are identified. The knowledge of the development of the resolution of the task and the effects and causes of each notion involved in the process arise (Beaney, 2018; Peckhaus, 2002).

- *Decomposition*. According to Plato and Pappus, this kind of reasoning allows for the reduction of the problem to its simplest components. The properties that define the assignment and the relationships between the most complex and the simplest objects involved in it are identified by extracting and investigating the principles that are at the base of the task. Aristotles, for example, underlines the fact that "sometimes, to solve a geometrical problem, you can only analyse a figure", breaking it down into its basic components and understanding the different parts of it (Beaney, 2018).

- *Introduction of auxiliary elements*. Kant, Polya and Hintikka, focus their attention on a fundamental process part: the introduction of new elements (a known Geometry practice: the auxiliary constructions). In the progressive and deductive processes all the bases are given and from these, consequences are elaborated. Unlike the backward reasoning, new notions appear and develop throughout the resolution at specific moments, according to the solver needs (Beaney, 2018; Hintikka & Remes, 1974).

EPISTEMIC ACTIONS

The concept of epistemic action was introduced into cognitive sciences by Kirsh and Maglio (1994) to indicate those physical actions that facilitate cognition and allow problems to be solved more quickly. These actions help to acquire useful information for the resolution that are hidden or difficult to compute mentally, have the purpose to simplify the mental processes. In mathematics education the term was first used by Hershkowitz, Schwarz y Dreyfus, (2001), who derived it from Pontecorvo and Girardet (1993) in their research on abstraction. The mental processes that occur in the student when solving a problem are not directly observable but can be identified through the analysis of the students' verbalisation or their physical actions. Epistemic actions are those actions that allow to identify the mental progresses in which knowledge is used or built and to operationally describe the procedures. They develop within the argumentative processes and are the basis of the interpretative activities. The actions involve procedures of a high methodological and metacognitive level and include the explanation of those procedures used for the interpretation of particular events.

The research of the last twenty years has resulted in the development of the theory of Abstraction in Context (AiC) (Dreyfus and Kidron, 2014) which aims to provide a theoretical and methodological approach, at the micro level, on the processes of learning mathematical knowledge. From a theoretical point of view, AiC attempts to create a bridge between cognitive knowledge and theories of abstraction, constructivist theory and the theory of activity. From a methodological point of view, AiC is a tool that allows the analysis of thought processes. The central theoretical construct of AiC is a theoretical-methodological model, according to which the emergence of a new construct is described and analyzed by means of three observable epistemic actions: recognizing (R), building-with (B), and constructing (C).

- Recognizing. It consists in recognizing some previously learned knowledge as relevant for the resolution of the problem.

- Building-with. It consists of combining a set of knowledge with the aim of achieving a specific objective. Objectives can be: to implement a strategy, to meet a justification to a conjecture, to find the solution to the problem.

- Constructing. It consists in assembling and integrating the previous knowledge with the aim of producing a new construct.

RESEARCH AIM AND METHODOLOGY

Aim

The aim of this paper is to show the use of AiC combined with the characterization of backward reasoning as a tool for the analysis of the epistemic actions involved in discovery processes. Both epistemic and cognitive elements are highlighted to examine how university students develop backward reasoning.

Participants and instrument

Data were collected in 2018 from 185 Spanish and Italian mathematics students, aged between 19 and 30. The participants students are spread all over all the years of academic studies from the first year of Bachelor to the last year of PhD (Table 1). They have different mathematical notions with regard to solving problems, but they had not received any special training about backtracking heuristics. These data are summarised in Table 1.

Mathematics Bachelor Italy	Mathematics Bachelor Spain	Future High School teachers (Master-students)	Mathematics PhD	Total
99	50	28	8	185

Table 1. Participants

To study the epistemic and cognitive characterization of backward reasoning in strategy games resolution we choose the 3D Tic-Tac-Toe (Golomb and Hales, 2002). This is a finite 2 players game with perfect information. Generally, it is played with paper and pencil. The board of the k-dimensional Tic-Tac-Toe ($k>1$) is a k-dimensional cube of side n, i.e. a (n, k)-board. The two players choose to adopt "X" or "O" to indicate the position of their pawns on the board. The game version used for the this research project experiments consists of a (4,3)-board. The board was presented in its two-dimensional representation (Fig. 1). The objective is to place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.

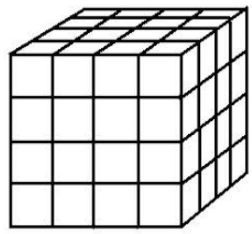
The given task (Fig. 1) consists in solving the game and finding a relationship between the number of winning lines and the board dimensions. Some mathematical notions acquired in university degree are necessary to solve it.

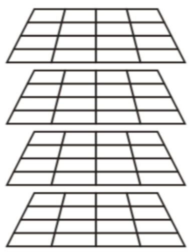
The methods for obtaining the data are direct observations during the working session, the recordings from the cameras, and the documents where students describe their approaches to the problem solution on protocols. The students worked in pairs or alone; we gave each pair of students paper and pencil and some “empty board”, using which they could elaborate a game strategy. Students were also asked to describe their approaches to solving the problem specifically describing: their thought processes in the resolution, the difficulties they encountered, and the strategies they would use in order to solve with paper and pencil. Students had two hours to do that.

3D Tic-Tac-Toe is the three dimensional version of the classic Three in a Skate game. The game board is a 4x4x4 cube.

The game is for two players. One player uses "crosses" and the other uses "zeros".

The objective is to place 4 marks in a row horizontally, diagonally or vertically while trying to block the opponent from doing so.





1. By helping yourself with the two-dimensional version of the game board, solve the game by developing your thinking process with a detailed solution protocol.

2. Mathematically express (formula, pattern, routine, ...) the relationships that can happen between the dimensions of the game board and the winning lines

Figure 1: Strategy game statement

A qualitative analysis was chosen to examine the resolution protocols of the students through the combination of the Backward Reasoning Epistemic Model and the AiC Model. We will illustrate it through a significant example in next section.

RESULTS: CASE STUDY

In this section we analyse a single student's resolution protocol of the 3D Tic-Tac-Toe. This allows us to get a deep understanding of the tendencies of the behaviour related to the sequences of actions during the discovery phase of resolution. The chosen student, whom we name A, is key informant of the PhD students group. A is an expert student, who solved the problem by investigating the mathematical relationships that are at the basis of this game using backward reasoning.

The student begins the game resolution by solving the 2D version of the game (3x3 board). First, he plays trying to remember the winning strategy, then he starts calculating mathematically the number of winning lines. Then he moves on to the 3D version of the game where he continues to reason about the number of winning lines until he obtains a general formula. Then it shows that the formula that he has found is valid for any cube of dimension (n, d) and finally he reasons again about the winning strategy, this time for the 3D case. The extract refers to the discovery process that the student makes to formulate the general recursive formula that allows to identify the number of winning lines knowing the size of the game board. Backward reasoning is predominant in this excerpt (Fig. 2).

12. I decide to move on to the 3D case. The previous strategy suggests me to count lines. I make a few drawings to test. There are 10 lines in each plan parallel to the axes and there are 12 planes parallel to the axes. I lack the “diagonal lines” as in the example. They seem more complicated.
13. I'm starting to do numerology: $10 = 4 * 2 + 2$ which is broken down as the number of pawns per dimension of the plane plus two diagonals. Will it be general?
14. I realize that $12 = 4 * 3$ that seems to follow the previous pattern. Hope. It looks like a nice combinatorial problem.
15. It reminds me of geometry calculations on finite fields. I think about shooting over there, but I realize that there are cyclic lines that come out on one side and appear on the opposite side. These movements are not allowed. I could rule them out but it seems too complicated. I abandon this strategy.
16. I think of a recursive pattern. I guess n pieces in d dimensions (the usual case is $(n, d) = (3,2)$ and this is $(n, d) = (4,3)$). Maybe the number of straight lines follows a pattern.
$$L(n, d) = cnt(n, d) * L(n, d - 1) + Diagonals$$
17. The constant must be the number of planes parallel to the axes. As in the previous case, these have to be nd, then I refine my formula to
$$L(n, d) = nd * L(n, d - 1) + Diagonals$$
18. Diagonals don't seem that simple. I start to play with the example of the cube and the plane. They seem to join opposite vertices of opposite faces. Will it be general?
19. I calculate that a hypercube has 2^d vertices, which gives me two faces with 2^{d-1} vertices. Thus, if my previous observation is correct, the formula is
$$L(n, d) = nd * L(n, d - 1) + 2^{d-1}$$

Fig. 2: Extract of student protocol

The student begins the resolution of the case in 3 dimensions thinking in analogy with the resolution of the case in 2 dimensions that he has previously carried out. The first objective is to count the winning lines on the board. To do so, he divides the game board into planes and counts the winning lines present on each plan. He then begins to think about the number of lines in each floor and breaks it down trying to identify the parts of the number with elements of the game (number of checkers for each winning line, size, number of diagonals). He then analyse and decompose each floor in the same way. At this point he introduces a recursive "auxiliary pattern" and conjectures the existence of a general recursive formula that relates the number of winning lines with the size of the table. He then analyses the formula and looks for a mathematical expression for each part of it. He then obtains the general recursive formula.

Analysing the extract, it is possible to identify different epistemic actions performed by the student. Using as definition of epistemic action: "that action in which knowledge is used or constructed". Each epistemic action can be characterized as an expression of the different characteristics of backward reasoning: in this extract we can see elements of decomposition (D) and insertion of auxiliary elements (E) and solution formulation (FS). In the same way, the same action can be classified according to the AiC model.

In the table below the second column identify the actions, the third identify the characteristics of backward reasoning and the last identify the AiC classification.

Protocols	Epistemic action	BR	AiC
12	Splitting the game board into planes Counting the winning lines in each plan Grouping winning Lines into a Scheme	D D E	B B R
13	Mathematically break down a number Identify each element of the decomposition	D E	C C
14	Mathematically break down a number	D	B
15	Analogy/ break motion		
16	Introduce a recursive pattern Conjecture: general recursive formula	E FS	R C
17	Break down the formula into its elements Analyse the constant element	D	C
18	Analyse the diagonal element	D	B
19	Representation of the diagonal in relation to the vertices of the hypercube Formulation of the general formula	D FS	C B

Table 2: Analysis of Epistemic action

Analysing the epistemic actions from the point of view of backward reasoning, one can observe how the student breaks down the problem and inserts auxiliary elements in an alternating way in order: first to conjecture the existence of a general formula and then to represent it mathematically. From the point of view of the analysis with AiC-model one can notice a certain regularity in the alternation of the AE (Table 2): Two sequences B-R-C-B-R-C characterize the formulation of the conjecture, while two sequences C-B-C-B characterize the formulation of the general formula. The actions that characterize the "decomposition" are actions that do not develop instantaneously in the resolution process but that suppose a longer time of realization. If you look at the introduction of auxiliary elements, these actions are instead instantaneous. Some actions, such as the introduction of a recursive pattern, can be a recognition of concepts belonging to the student's background, it happens after a structural analogy. During this analogy (line 15) the student remembers geometrical concepts that help him to identify patterns. In other actions, such as the identification of each element of the decomposition of the number 10 with an element of the game, the student creates a new construct from the processing of knowledge already encountered in the resolution.

CONCLUSION

In this investigation, we chose to use the AiC - model to understand how the process of abstraction develops in the construction of new mathematical knowledge using backward reasoning. The development of the different epistemic actions was analysed, with the help of the subcategories built in this research and the relationships they established among themselves. If we look at the whole process of the protocol (Fig. 3), we can see how the student passes through different contexts in order to achieve the general mathematical formulation. He begins working within the game context, then he moves to a mathematical context to interpret the example through this new lens, then he goes forward and explains the game in a more general mathematical context.

The transition between the three contexts happens with a complex back and forth process, where the different contexts are repeatedly activated, as illustrated in Fig. 3.

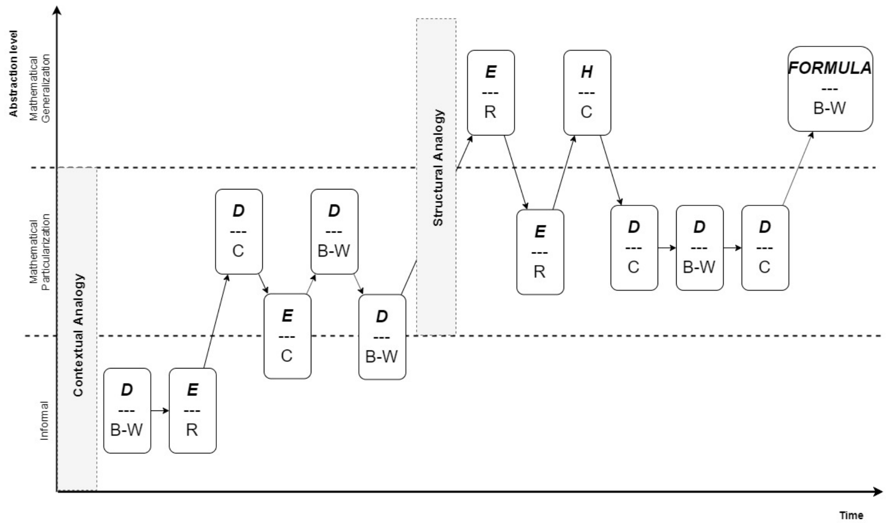


Fig. 3 Pattern in epistemic actions and context

Following the introduction of subcategories of analysis, built in a narrow link with the nature of backward reasoning it is possible to analyse, in detail, characteristics associated with the development of students' thinking processes. This helps in better understanding the connections between the different epistemic actions that can influence Building with and Construction.

We notice that the incorporation of Backward Reasoning-based categories allows to identify breaking elements and which they trigger the construction process of the formula. These actions occur and they cannot be determined only with specific actions defined according to the AiC-model. In this case it has been necessary to identify elements of the constructive epistemic action processes produced in the long term. The back and forth movement above is identified as a cognitive travel between the concrete and the abstract: in it the analogy processes — both contextual and structural analogy— have been crucial. In the process of conjecturing and justifying, complex chains of plausible reasoning are often elaborated, which may contain new nuances that enrich already known patterns. An exhaustive analysis of these processes requires an exploration not only of the punctual but in the long term.

Acknowledgment: This study was funded by *Interdisciplinary Mathematics Institute* (IMI) special action grant, Programme INVEDUMAT_uni (Research in Mathematical Education at

University Level), Spain and Science Doctoral School of Università and Politecnico of Torino.

REFERENCES

Antonini, S. (2011). Generating examples: focus on processes. *ZDM Mathematics Education*, 43: 205–217.

Beaney, M. (2018). Analysis. In *The Stanford Encyclopedia of Philosophy*. Edward N. Zalta (ed.). Metaphysics Research Lab, Stanford University.

Dreyfus, T. y Kydron. I. (2014). Introduction to Abstraction in Context (AiC). In: Bikner-Ahsbahs, A., y Prediger, S. (Eds). *Networking of Theories as a Research Practice in Mathematics Education*. (pp. 85-96). Heidelberg: Springer.

Golomb, S.W., Hales, A.W. (2002). Hypercube Tic-Tac-Toe, in: More Games of No Chance, In R.J. Nowakowski (Ed.) *Proc. MSRI Workshop on Combinatorial Games*, July 2000, Berkeley, CA. MSRI Publ. Vol.42, (pp.167-182). Cambridge: Cambridge University Press.

Barbero, M, Gómez-Chacón, IM., Arzarello, F. (2020). Backward Reasoning and Epistemic Actions in Discovering Processes of Strategic Games Problems. *Mathematics*. 8(6): 989. https://doi.org/10.3390/math8060989

Barbero, M. & Gómez-Chacón, I. M. (2018). Analyzing Regressive reasoning at university level. In V. Durand-Guerrier, et.al. (Eds), *Proceedings of the Second INDRUM*, (pp. 204-213). Kristiansand, Norway: University of Agder.

Gómez-Chacón, I. M. & Barbero, M. (2019). Situation Specific-Skills Working Backward Reasoning: The Student's Perceptions and Affect Dimension, *CERME11 proceedings*.

Gómez-Chacón, I. M. (2017). Emotions and heuristics: the state of perplexity in mathematics. *ZDM-International Journal on Mathematics Education*, 49, 323-338.

Hershkowitz, R., Schwarz, B. B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*. 32, 195-222.

Hintikka, J., Remes, U. (1974). *The Method of Analysis: Its Geometrical Origin and Its General Significance.* Reidel, Dordrecht.

Kirsh, D., Maglio, P. (1994). On distinguishing epistemic from pragmatic action. *Cognitive Science* 18, 513-549.

Peckhaus, V. (2000). *Regressive Analysis*. University Essen. Forschungsbericht.

Polya, G. (1945). *How to Solve it: A New Aspect of Mathematical Method*. Doubleday, New York.

Pontecorvo and Girardet (1993). Arguing and reasoning in understanding historical topics. *Cognition and Instruction*, 11, 365-395.

Wickelgren, W. (1974). *How to solve problems. Elements of a theory of problems and problem solving*. San Francisco: Freeman and Company.