Extension of the 2-p-opt and 1-shift Algorithms to the Heterogeneous Probabilistic Traveling Salesman Problem*†

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Abstract

The probabilistic traveling salesman problem is a well known problem that is quite challenging to solve. It involves finding the tour with the lowest expected cost given that customers will require a visit with a given probability. There are several proposed algorithms for the homogeneous version of the problem, where all customers have identical probability of being realized. From the literature, the most successful approaches involve local search procedures, with the most famous being the 2-p-opt and 1-shift procedures proposed by Bertsimas [3]. Recently, however, evidence has emerged that indicates the equations offered for these procedures are not correct, and even when corrected, the translation to the heterogeneous version of the problem is not simple. In this paper we extend the analysis and correction to the heterogeneous case. We derive new expressions for computing the cost of 2-p-opt and 1-shift local search moves, and we show that the neighborhood of a solution may be explored in $O(n^2)$ time, the same as for the homogeneous case, and not in $O(n^3)$ as first reported in the literature.

Keywords: probabilistic traveling salesman, heuristics, local search, stochastic vehicle routing

1 Introduction

For many companies, only a subset of their customers require a pickup or delivery each day. Information may be not available far enough in advance to create optimal schedules each day for those customers that do require a visit or the cost to acquire sufficient computational power to find such solutions may be prohibitive.

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For these reasons, it is not unusual to design a distance minimizing tour containing all customers, and each day follow the ordering of this a priori tour to visit only the realized customers. The problem of finding an a priori tour τ of minimum cost in an expected value sense, given a set of customers $\{i \mid i = 1, 2, ..., n\}$ with probabilities $\{p_i \mid i = 1, 2, ..., n\}$ of requiring a visit, defines the Probabilistic Traveling Salesman Problem (PTSP). The objective of the PTSP is to minimize the expected length of the a priori tour over all customer subsets.

Jaillet [7] was the first to look at how to evaluate the expected value of an a priori tour where the number of customers that require a visit is determined according to a known probability distribution. In [7, 8], Jaillet points out that the optimal TSP tour through a set of customers is often not the optimal tour in an expected value sense which means that the PTSP should be solved separately from the TSP.

Due to the probabilistic nature of the problem, the cost of evaluating a proposed solution for the PTSP is expensive. It is quite challenging to solve such problems optimally even when they are solved a priori where time is less of an issue. An exact algorithm is described in [9], but it has been applied primarily to small instances of the problem. Most approaches in the PTSP literature focus on heuristics that efficiently find good, but not necessarily optimal, solutions (see for instance [3, 4] and the references cited therein). One crucial ingredient in these heuristic approaches is the design of an effective local search algorithm, which makes small moves in the solution space trying to improve the solution quality. In the PTSP, the use of an expected value-based cost to evaluate a local search move, rather than a standard TSP local search procedure, grows increasingly important as the number of customers increases [3]. Thus, it is critical that the expected value-based costs in the local search procedures are quick to evaluate.

In the literature there are two local search procedures created specifically for the PTSP that evaluate a change in terms of expected value: the 2-p-opt and 1-shift. The 2-p-opt is the probabilistic version of the famous 2-opt procedure created for the TSP [10] which evaluates the change in distance traveled if the sequence between two points in the tour is reversed. The 1-shift is the evaluation of the change in expected value associated with removing a customer from the tour and inserting it at another point in the tour.

For PTSP instances where each customer is present with the same probability (the homogeneous PTSP), Bertsimas proposed move evaluation expressions in [3] that explore the neighborhood of a solution (that is, that verify whether an improving 2-p-opt or 1-shift move exists) in $O(n^2)$ time. The intent of Bertsimas' equations is to provide a recursive means to quickly compute the exact change in expected value associated with either a 2-p-opt or 1-shift procedure. Evaluating the cost of a local move by computing the cost of two neighboring solutions and then evaluating their difference would require much more time ($O(n^4)$) than a recursive approach. Recently Bianchi et al.[5] re-analyzed and corrected Bertsimas' expressions, after evidence emerged that they did not exactly evaluate the cost of a 2-p-opt and 1-shift move. The correction of these equations confirms that it is possible to explore both the 2-p-opt and 1-shift neighborhood of a solution in $O(n^2)$ time, and does, as expected, create significant improvement in the already good results for the homogeneous PTSP.

The heterogeneous problem, where probabilities at the various customers are allowed to vary, is actually a more important problem because it is clearly closer to real world applications. As companies gather and retain more information about their customers, heterogeneous probabilities will become increasingly available in practice. Few of the results in the literature apply, though, when probabilities are not homogeneous. One paper, [1], provides a lower bound for the heterogeneous PTSP, and another paper, [2], reports computational results of 2-p-opt and 1-shift local search algorithms applied to some small heterogeneous PTSP instances. The results in [2] are based on the work of Chervi [6], who proposed recursive expressions for the cost of 2-p-opt and 1-shift moves for the heterogeneous PTSP. Chervi's expressions explore the 2-p-opt and 1-shift neighborhoods in $O(n^3)$ time, suggesting that it is not possible to retain the $O(n^2)$ complexity of the homogeneous PTSP. Moreover, Chervi's expressions reduce to the incorrect expressions for the homogeneous

PTSP published in [3], when all customer probabilities are equal, and therefore are also not correct.

In this paper, we extend the analysis performed in [5] for the homogeneous case to the heterogeneous case and derive new expressions for computing the cost of 2-p-opt (section 3) and 1-shift (section 4) local search moves. We demonstrate that the neighborhood of a solution for this important problem may be explored in $O(n^2)$ time, thus retaining the same complexity as the homogeneous case.

2 Notation and objective function

Throughout the paper we use the following notation. Without loss of generality we consider the a priori tour $\tau = (1, 2, \cdots, n)$ with

$$i \pm j = \begin{cases} (i \pm j) \bmod n & \text{iff } (i \pm j) \neq 0 \text{ and } (i \pm j) \neq n \\ n & \text{otherwise,} \end{cases} \tag{1}$$

where $b(\text{mod } n) = a$ if the difference $b - a$ is evenly divisible by n. Given the probability p_i that customer i requires a visit, we define $q_i = 1 - p_i$ as the probability that i does not require a visit, and

$$\prod_i^j q = \begin{cases} \prod_{t=i}^j q_t & \text{if } i \leq j \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

The expected length of a priori tour $\tau = (1, 2, ..., n)$ can be computed in $O(n^2)$ time with the following expression [7]

$$E[L(\tau)] = \sum_{i=1}^{n} \sum_{r=1}^{n-1} d(i, i+r) p_i p_{i+r} \prod_{t=i+1}^{i+r-1} q. \tag{3}$$

Each term in the summation described by (3) represents the distance between the i^{th} city and the $(i+r)^{\text{th}}$ city weighted by the probability that the two nodes require a visit ($p_i p_{i+r}$) while the $r-1$ nodes between them do not require a visit ($\prod_{t=i+1}^{i+r-1} q$).

It is convenient here to introduce the following two dimensional matrices of partial results A and B that will be used as building blocks of the 2-p-opt and 1-shift evaluation expressions:

$$A_{i,k} = \sum_{r=k}^{n-1} d(i, i+r) p_i p_{i+r} \prod_{i+1}^{i+r-1} q \tag{4}$$

$$B_{i,k} = \sum_{r=k}^{n-1} d(i-r, i) p_{i-r} p_i \prod_{i-r+1}^{i-1} q, \tag{5}$$

with $1 \leq k \leq n-1$ and $1 \leq i \leq n$. The matrices A and B are straightforward extensions to the heterogeneous PTSP of corresponding matrices defined for the homogeneous case (see [5]).

3 The 2-p-opt: derivation of an efficient cost evaluation expression

For an a priori tour τ, its 2-p-opt neighborhood is the set of tours obtained by reversing a section of τ (that is, a set of consecutive nodes) such as the example in Figure 1. Consider, without loss of generality, a tour

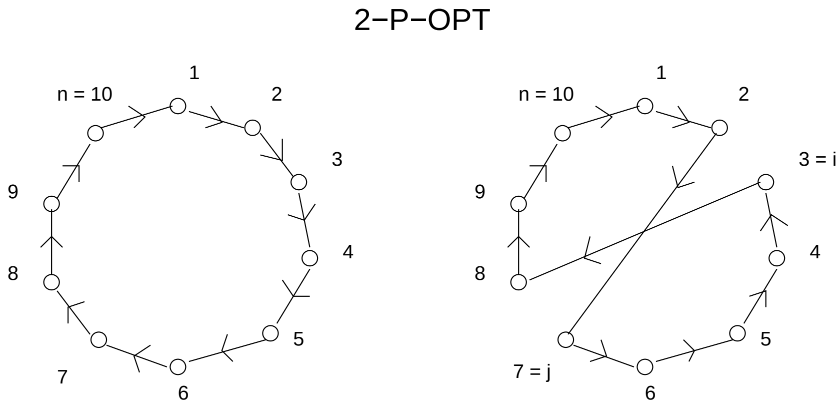

Figure 1: Tour $\tau = (1, 2, ..., i, i+1, ..., j, j+1, ..., n)$ (left) and tour $\tau_{i,j} = (1, 2, ..., i-1, j, j-1, ..., i, j+1, ..., n)$ (right) obtained from τ by reversing the section $(i, i+1, ..., j)$, with $n = 10$, $i = 3$, $j = 7$.

$\tau = (1, 2, ..., i, i+1, ..., j, j+1, ..., n)$ and a tour $\tau_{i,j} = (1, 2, ..., j, j-1, ..., i, j+1, ..., n)$ obtained by reversing a section $(i, i+1, ..., j)$ of τ, where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n\}$, and $i \neq j$. Let $\Delta E_{i,j}$ denote the change in the expected tour length $E[L(\tau_{i,j})] - E[L(\tau)]$. We will derive a set of recursive formulas for $\Delta E_{i,j}$ that can be used to efficiently evaluate a neighborhood of 2-p-opt moves. To describe this procedure, we will use the following terms and notation:

Definition 1 *$inside_{i,j} = \{i, \ldots, j\}$, the nodes 'inside' the reversed section of $\tau_{i,j}$*

Definition 2 *$outside_{i,j} =$ all nodes $\setminus$ $inside_{i,j}$*

Definition 3 *$\Delta E_{i,j}\big|_{S \to T} =$ the contribution to $\Delta E_{i,j}$ due to the arcs from the nodes in S to the nodes in T.*

Definition 4 *$\Delta E_{i,j}\big|_{S \leftrightarrow T} =$ the contribution to $\Delta E_{i,j}$ due to the arcs from the nodes in S to the nodes in T and from the nodes in T to the nodes in S.*

Unlike the TSP, the expected cost of an a priori tour involves all of the arcs between nodes. The ordering of the nodes on the a priori tour affects the probability of an arc being used, and this probability determines the weight on the arcs. The change in expected tour length, $\Delta E_{i,j}$, resulting from a reversal of a section is thus based on the change in weight placed on certain arcs in the two tours τ and $\tau_{i,j}$. It is not difficult to verify that, as in the homogeneous PTSP, the weight placed on arcs between inside nodes $inside_{i,j}$ and between outside nodes $outside_{i,j}$ do not change as a result of the reversal. In other words, the contributions to $\Delta E_{i,j}$ due to $\Delta E_{i,j}\big|_{inside_{i,j} \leftrightarrow inside_{i,j}}$ and to $\Delta E_{i,j}\big|_{outside_{i,j} \leftrightarrow outside_{i,j}}$ are zero. Therefore, $\Delta E_{i,j}$ is based on the change in weight placed on arcs between $inside_{i,j}$ and $outside_{i,j}$, that is,

$$\Delta E_{i,j} = \Delta E_{i,j}\big|_{inside_{i,j} \leftrightarrow outside_{i,j}}. \tag{6}$$

The contribution to $\Delta E_{i,j}$ due to arcs between inside and outside may be split into three pieces:

$$\Delta E_{i,j}\big|_{inside_{i,j} \leftrightarrow outside_{i,j}} = E^{(1)}_{i,j} + E^{(2)}_{i,j} - E^{(3)}_{i,j}, \tag{7}$$

where the three terms on the right hand side of the last equation are, respectively, the contribution to $E[L(\tau_{i,j})]$ due to the arcs going from $inside_{i,j}$ to $outside_{i,j}$, the contribution to $E[L(\tau_{i,j})]$ due to the arcs going from $outside_{i,j}$ to $inside_{i,j}$, and the contribution to $E[L(\tau)]$ due to arcs joining the two customer sets in both directions. In terms of our defined notation, we can express the three contributions as follows

$$E^{(1)}_{i,j} = E[L(\tau_{i,j})]\big|_{inside_{i,j} \to outside_{i,j}} \tag{8}$$

$$E^{(2)}_{i,j} = E[L(\tau_{i,j})]\big|_{outside_{i,j} \to inside_{i,j}} \tag{9}$$

$$E^{(3)}_{i,j} = E[L(\tau)]\big|_{inside_{i,j} \leftrightarrow outside_{i,j}}. \tag{10}$$

Unlike the TSP, there is an expected cost associated with using an arc in a forward direction as well as a reverse direction, and these costs are usually not the same. The costs are based on which customers would have to be "skipped" in order for the arc to be needed in the particular direction. For example, the weight on arc $(1, 2)$ is based only on the probability of nodes 1 and 2 requiring a visit, where the weight on arc $(2, 1)$ is also based on nodes $(2, 3, \cdots, n)$ not requiring a visit. For the homogeneous PTSP, the equations are much simpler since the only consideration is how many nodes are skipped, not which arcs are skipped.

We will now derive recursive expressions for $E^{(1)}_{i,j}$, $E^{(2)}_{i,j}$, $E^{(3)}_{i,j}$, respectively, in terms of $E^{(1)}_{i+1,j-1}$, $E^{(2)}_{i+1,j-1}$ and $E^{(3)}_{i+1,j-1}$. Here, we assume that $j = i + k$ and $k \geq 2$. Let us consider the tour $\tau_{i+1,j-1} = (1, 2, ..., i - 1, i, j - 1, j - 2, ..., i + 1, j, j + 1..., n)$ obtained by reversing a section $(i + 1, ..., j - 1)$ of τ. We can make three important observations. The first one is that the partitioning of customers with respect to $\tau_{i,j}$ is related to the partitioning with respect to $\tau_{i+1,j-1}$ in the following way:

$$inside_{i,j} = inside_{i+1,j-1} \cup \{i, j\} \tag{11}$$

$$outside_{i,j} = outside_{i+1,j-1} \setminus \{i, j\}. \tag{12}$$

The second observation is that for any arc (l, r), with $l \in inside_{i+1,j-1}$ and $r \in outside_{i,j}$, the weight on the arc in the expected total cost equation for $\tau_{i,j}$ can be obtained by multiplying the weight that the arc has in $\tau_{i+1,j-1}$ by q_i and dividing it by q_j. In order to see this, one should compare the set of skipped customers in $\tau_{i,j}$ with the set of skipped customers in $\tau_{i+1,j-1}$. In $\tau_{i,j}$, the fact that arc (l, r) is used implies that customers $(l - 1, l - 2, \ldots, i + 1, i, j + 1, \ldots, r - 1)$ are skipped, while in $\tau_{i+1,j-1}$ using arc (l, r) implies that customers $(l - 1, l - 2, \ldots, i + 1, j, j + 1, \ldots, r - 1)$ are skipped. Therefore, the set of skipped customers in $\tau_{i+1,j-1}$ is equal to the set of skipped customers in $\tau_{i,j}$ except for customer j in $\tau_{i+1,j-1}$, which is replaced by customer i in $\tau_{i,j}$. In terms of probabilities, our second observation can be expressed as

$$E[L(\tau_{i,j})]\big|_{inside_{i+1,j-1} \to outside_{i,j}} = \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to outside_{i,j}}. \tag{13}$$

The third important observation is similar to the previous one, but it refers to arcs going in the opposite direction. More precisely, for any arc (r, l), with $r \in outside_{i,j}$ and $l \in inside_{i+1,j-1}$, the weight of the arc in $\tau_{i,j}$ can be obtained by multiplying the weight that the arc has in $\tau_{i+1,j-1}$ by q_j and by dividing it by q_i. It is not difficult to verify this using the same argument as in the previous observation. Similar to the second observation, the third observation can be expressed as

$$E[L(\tau_{i,j})]\big|_{outside_{i,j} \to inside_{i+1,j-1}} = \frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]\big|_{outside_{i,j} \to inside_{i+1,j-1}}. \tag{14}$$

Now, by equations (8), (9) and (11) we can write

$$E_{i,j}^{(1)} = E[L(\tau_{i,j})]\big|_{inside_{i+1,j-1} \to outside_{i,j}} + E[L(\tau_{i,j})]\big|_{\{i,j\} \to outside_{i,j}} \tag{15}$$

$$E_{i,j}^{(2)} = E[L(\tau_{i,j})]\big|_{outside_{i,j} \to inside_{i+1,j-1}} + E[L(\tau_{i,j})]\big|_{outside_{i,j} \to \{i,j\}}. \tag{16}$$

By combining equation (13) with equation (15), we obtain

$$E_{i,j}^{(1)} = \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to outside_{i,j}} + E[L(\tau_{i,j})]\big|_{\{i,j\} \to outside_{i,j}}, \tag{17}$$

which, by equation (12) becomes

$$\begin{aligned} E_{i,j}^{(1)} =& \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to outside_{i+1,j-1}} \\ &- \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to \{i,j\}} + E[L(\tau_{i,j})]\big|_{\{i,j\} \to outside_{i,j}}. \end{aligned} \tag{18}$$

We can rewrite this to obtain the following recursion

$$E_{i,j}^{(1)} = \frac{q_i}{q_j} E_{i+1,j-1}^{(1)} - \frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to \{i,j\}} + E[L(\tau_{i,j})]\big|_{\{i,j\} \to outside_{i,j}}. \tag{19}$$

In an analogous way, we can create a recursive expression for $E_{i,j}^{(2)}$. By first combining equation (13) with equation (16), and then applying (12), we obtain

$$E_{i,j}^{(2)} = \frac{q_j}{q_i} E_{i+1,j-1}^{(2)} - \frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]\big|_{\{i,j\} \to inside_{i+1,j-1}} + E[L(\tau_{i,j})]\big|_{outside_{i,j} \to \{i,j\}}. \tag{20}$$

Let us now focus on $E_{i,j}^{(3)}$ (equation (10)). This term refers to the original tour τ. Therefore, in order to get a recursive expression in terms of $E_{i+1,j-1}^{(3)}$, we must isolate the contribution to $E_{i,j}^{(3)}$ due to arcs going from $inside_{i+1,j-1}$ to $outside_{i+1,j-1}$ and vice versa. Thus, by combining equation (10) with both (11) and (12) we obtain

$$E_{i,j}^{(3)} = E_{i+1,j-1}^{(3)} - E[L(\tau)]\big|_{\{i,j\} \leftrightarrow inside_{i+1,j-1}} + E[L(\tau)]\big|_{\{i,j\} \leftrightarrow outside_{i,j}}. \tag{21}$$

Now we complete the derivation by showing that it is possible to express the 'residual' terms on the right hand side of $E_{i,j}^{(s)}$, $s = 1, 2, 3$ in (19), (20), (21) in terms of the matrices A (equation (4)) and B (equation (5)).

Recalling that $j = i + k$, the second term on the right hand side of equation (19) is the following

$$\begin{aligned} -\frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to \{i,j\}} =& -\frac{q_i}{q_j} \sum_{t=1}^{k-1} d(i+t, i) p_{i+t} p_i \prod_{i+1}^{i+t-1} q \prod_{i+k}^{i+n-1} q \\ &- \frac{q_i}{q_j} \sum_{t=1}^{k-1} d(i+t, i+k) p_{i+t} p_{i+k} \prod_{i+1}^{i+t-1} q, \end{aligned} \tag{22}$$

which, by applying the definition of A (4), becomes

$$-\frac{q_i}{q_j} E[L(\tau_{i+1,j-1})]\big|_{inside_{i+1,j-1} \to \{i,j\}} = -\frac{q_i}{q_j} \left(\prod_{i+k}^{i+n-1} q \right) (A_{i,1} - A_{i,k}) - \left(\prod_{i+k}^{i+n-1} q \right)^{-1} A_{i+k,n-k+1}. \tag{23}$$

The rightmost term of equation (19) may be written as

$$E[L(\tau_{i,j})]\Big|_{\{i,j\}\to outside_{i,j}} = \sum_{r=1}^{n-k-1} d(i, i+k+r) p_i p_{i+k+r} \prod_{i+k+1}^{i+k+r-1} q + \sum_{r=1}^{n-k-1} d(i+k, i+k+r) p_{i+k} p_{i+k+r} \prod_{i+k+1}^{i+k+r-1} q \prod_{i}^{i+k-1} q. \tag{24}$$

By applying the definition of A (4) to the right hand side of the last equation we obtain

$$E[L(\tau_{i,j})]\Big|_{\{i,j\}\to outside_{i,j}} = \frac{q_i}{q_j}\left(\prod_{i}^{i+k-1} q\right)^{-1} A_{i,k+1} + \left(\prod_{i}^{i+k-1} q\right)(A_{i+k,1} - A_{i+k,n-k}). \tag{25}$$

The second term on the right hand side of equation (20) is the following

$$-\frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]\Big|_{\{i,j\}\to inside_{i+1,j-1}} = -\frac{q_j}{q_i}\sum_{t=1}^{k-1} d(i, i+k-t) p_i p_{i+k-t} \prod_{i+k-t+1}^{i+k-1} q - \frac{q_j}{q_i}\sum_{t=1}^{k-1} d(i+k, i+k-t) p_{i+k} p_{i+k-t} \prod_{i+k-t+1}^{i+k-1} q \prod_{i-n+k+1}^{i} q, \tag{26}$$

which, by applying the definition of B (5), becomes

$$-\frac{q_j}{q_i} E[L(\tau_{i+1,j-1})]\Big|_{\{i,j\}\to inside_{i+1,j-1}} = -\frac{q_j}{q_i}\left(\prod_{i+k}^{i+n-1} q\right)^{-1} B_{i,n-k+1} - \left(\prod_{i+k}^{i+n-1} q\right)(B_{i+k,1} - B_{i+k,k}). \tag{27}$$

The rightmost term of equation (20) may be written as

$$E[L(\tau_{i,j})]\Big|_{outside_{i,j}\to \{i,j\}} = \sum_{r=1}^{n-k-1} d(i-r, i) p_{i-r} p_i \prod_{i-r+1}^{i-1} q \prod_{i+1}^{i+k} q + \sum_{r=1}^{n-k-1} d(i-r, i+k) p_{i-r} p_{i+k} \prod_{i-r+1}^{i-1} q. \tag{28}$$

By applying the definition of B (5) to the right hand side of the last equation we obtain

$$E[L(\tau_{i,j})]\Big|_{outside_{i,j}\to \{i,j\}} = \frac{q_j}{q_i}\left(\prod_{i}^{i+k-1} q\right)(B_{i,1} - B_{i,n-k}) + \left(\prod_{i}^{i+k-1} q\right)^{-1} B_{i+k,k+1}. \tag{29}$$

The second term on the right hand side of equation (21) is the following

$$
\begin{aligned}
-E[L(\tau)]\Big|_{\{i,j\}\leftrightarrow inside_{i+1,j-1}} = &-\sum_{t=1}^{k-1} d(i,i+t)p_i p_{i+t} \prod_{i+1}^{i+t-1} q \\
&-\sum_{t=1}^{k-1} d(i+k,i+t)p_{i+k}p_{i+t} \prod_{i-n+k+1}^{i+t-1} q \\
&-\sum_{t=1}^{k-1} d(i+k-t,i)p_{i+k-t}p_i \prod_{i+k-t+1}^{i+k-1} q \prod_{i+k}^{i+n-1} q \\
&-\sum_{t=1}^{k-1} d(i+k-t,i+k)p_{i+k-t}p_{i+k} \prod_{i+k-t+1}^{i+k-1} q,
\end{aligned}
\tag{30}
$$

which, by applying the definition of A (4) and B (5), becomes

$$
-E[L(\tau)]\Big|_{\{i,j\}\leftrightarrow inside_{i+1,j-1}} = -\left(A_{i,1} - A_{i,k} + A_{i+k,n-k+1} + B_{i,n-k+1} + B_{i+k,1} - B_{i+k,k}\right). \tag{31}
$$

The rightmost term of (21) is the following

$$
\begin{aligned}
E[L(\tau)]\Big|_{\{i,j\}\leftrightarrow outside_{i,j}} = &\sum_{r=1}^{n-k-1} d(i-r,i)p_{i-r}p_i \prod_{i-r+1}^{i-1} q \\
&+\sum_{r=1}^{n-k-1} d(i-r,i+k)p_{i-r}p_{i+k} \prod_{i-r+1}^{i+k-1} q \\
&+\sum_{r=1}^{n-k-1} d(i,i+k+r)p_i p_{i+k+r} \prod_{i+1}^{i+k+r-1} q \\
&+\sum_{r=1}^{n-k-1} d(i+k,i+k+r)p_{i+k}p_{i+k+r} \prod_{i+k+1}^{i+k+r-1} q,
\end{aligned}
\tag{32}
$$

which, by applying the definition of A (4) and B (5), becomes

$$
E[L(\tau)]\Big|_{\{i,j\}\leftrightarrow outside_{i,j}} = B_{i+1} - B_{i,n-k} + B_{i+k,k+1} + A_{i,k+1} + A_{i+k,1} - A_{i+k,n-k}. \tag{33}
$$

Now, by substituting in equations (19), (20) and (21) the appropriate terms from equations (25) ... (33), and by letting

$$
P_{i,j} = \prod_{i}^{j} q, \qquad \overline{P}_{i,j} = \prod_{j+1}^{i+n-1} q, \tag{34}
$$

we obtain the following final recursive equations for the 2-p-opt local search for $j = i + k$ and $k \geq 2$:

$$
\Delta E_{i,j} = E_{i,j}^{(1)} + E_{i,j}^{(2)} - E_{i,j}^{(3)}, \tag{35}
$$

$$E^{(1)}_{i,j} = \frac{q_i}{q_j} E^{(1)}_{i+1,j-1} + q_i P^{-1}_{i,j} A_{i,k+1} - q_i \overline{P}_{i,j}(A_{i,1} - A_{i,k}) - \frac{1}{q_j}\overline{P}^{-1}_{i,j} A_{j,n-k+1} + \frac{1}{q_j} P_{i,j}(A_{j,1} - A_{j,n-k}), \tag{36}$$

$$E^{(2)}_{i,j} = \frac{q_j}{q_i} E^{(2)}_{i+1,j-1} - \frac{1}{q_i}\overline{P}^{-1}_{i,j} B_{i,n-k+1} + \frac{1}{q_i} P_{i,j}(B_{i,1} - B_{i,n-k}) + q_j P^{-1}_{i,j} B_{j,k+1} - q_j \overline{P}_{i,j}(B_{j,1} - B_{j,k}), \tag{37}$$

$$E^{(3)}_{i,j} = E^{(3)}_{i+1,j-1} + A_{i,1} - A_{i,k} - A_{i,k+1} - (A_{j,1} - A_{j,n-k} - A_{j,n-k+1}) - (B_{i,1} - B_{i,n-k} - B_{i,n-k+1}) + B_{j,1} - B_{j,k} - B_{j,k+1}. \tag{38}$$

For $k = 1$, we can express the three contributions to $\Delta E_{i,i+1}$ (8), (9) and (10) in terms of A and B and obtain the following equations

$$E^{(1)}_{i,i+1} = \frac{1}{q_{i+1}} A_{i,2} + q_i(A_{i+1,1} - A_{i+1,n-1}) \tag{39}$$

$$E^{(2)}_{i,i+1} = q_{i+1}(B_{i,1} - B_{i,n-1}) + \frac{1}{q_i} B_{i+1,2} \tag{40}$$

$$E^{(3)}_{i,i+1} = A_{i,2} + A_{i+1,1} - A_{i+1,n-1} + B_{i,1} - B_{i,n-1} + B_{i+1,2}. \tag{41}$$

For $j = i$, $\Delta E_{i,i} = 0$ since $\tau_{i,i} = \tau$. It is still necessary, though, to compute the three contributions $E^{(s)}_{i,i}$, $s = 1, 2, 3$ separately, in order to compute the recursion $E^{(s)}_{i-1,i+1}$, $s = 1, 2, 3$. By expressing (8), (9) and (10) in terms of A and B one obtains

$$E^{(1)}_{i,i} = A_{i,1} \tag{42}$$

$$E^{(2)}_{i,i} = B_{i,1} \tag{43}$$

$$E^{(3)}_{i,i} = A_{i,1} + B_{i,1}. \tag{44}$$

Note that $\Delta E_{i,i} = E^{(1)}_{i,i} + E^{(2)}_{i,i} - E^{(3)}_{i,i} = 0$, as expected.

As an additional check on the validity of the above recursive expressions, it is possible to verify that when $p_i = p$ and $q_i = q = 1 - p$, we obtain the same recursive $\Delta E_{i,j}$ expressions of the homogeneous PTSP, as published in [5].

The 2-p-opt local search may proceed as with the homogeneous PTSP, the only difference being that in the heterogeneous case the three contributions to $\Delta E_{i,j}$ ($E^{(s)}_{i,j}$, $s = 1, 2, 3$) must be computed separately. The local search proceeds in two phases. The first phase consists of computing $\Delta E_{i,i+1}$ for every value of i (by means of equations (35) and (39)-(41)) while accumulating the two matrices of partial results A and B. Each time a negative $\Delta E_{i,i+1}$ value is encountered, the two nodes should immediately be switched. The n calculations of phase one require $O(n)$ time apiece, or $O(n^2)$ time in all. At the end of this phase, an a priori tour is reached for which every $\Delta E_{i,i+1}$ value is positive, and the matrices A and B are complete and correct for that tour. Additionally, at the end of the first phase, the matrices P and $\overline{P}$ (34) are computed (in $O(n^2)$ time), so they can be used in the second phase (in $O(1)$ time). The second phase of the local search consists of computing $\Delta E_{i,j}$ recursively by means of equations (35)-(38). Since each $\Delta E_{i,j}$ in phase two is computed in $O(1)$ time, this phase, and thus the entire 2-p-opt checking sequence, is performed in $O(n^2)$. With a straightforward implementation that does not utilize recursion, evaluating the 2-p-opt neighborhood would require $O(n^4)$ time instead of $O(n^2)$.

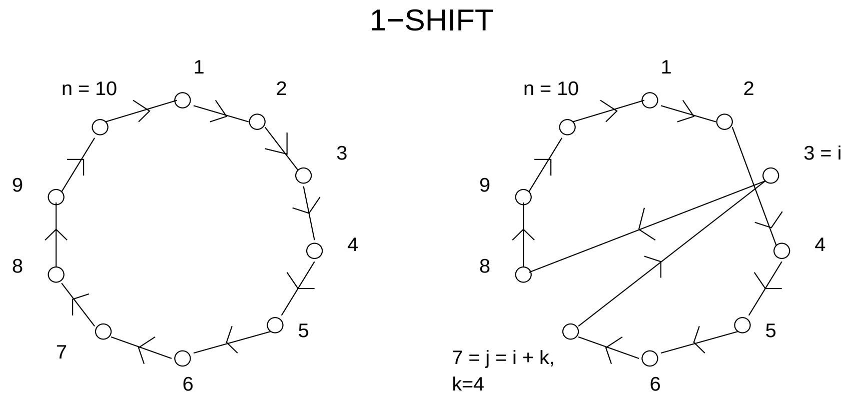

Figure 2: Tour $\tau = (1, 2, ..., i, i+1, ..., j, j+1, ..., n)$ (left) and tour $\tau_{i,j} = (1, 2, ..., i-1, i+1, i+2, ..., j, i, j+1, ..., n)$ (right) obtained from τ by moving node i to position j and shifting backwards the nodes $(i+1, ..., j)$, with $n = 10$, $i = 3$, $j = 7$.

4 The 1-shift: derivation of an efficient cost evaluation expression

Given an a priori tour τ, its 1-shift neighborhood is the set of tours obtained by moving a node which is at position i to position j of the tour, with the intervening nodes being shifted backwards one space accordingly, as in Figure 2.

Consider, without loss of generality, a tour $\tau = (1, 2, \ldots, i, i+1, \ldots, j, j+1, \ldots, n)$ and a tour $\tau_{i,j} = (1, 2, \ldots, i-1, i+1, \ldots, j, i, j+1, \ldots, n)$ obtained from τ by moving node i to position j and shifting backwards the nodes $i+1, ..., j$, where $i \in \{1, 2, \ldots, n\}$, $j \in \{1, 2, \ldots, n\}$, and $i \neq j$. Let $\Delta' E_{i,j}$ denote the change in the expected tour length $E[L(\tau_{i,j})] - E[L(\tau)]$. In the following, the correct recursive formula for $\Delta' E_{i,j}$ is derived for the 1-shift neighborhood. This derivation follows the same order as for the 1-shift for the homogeneous PTSP, which is described in detail in [5]. Therefore, we will only show the key features of the derivation here.

Let $j = i+k$. For $k = 1$, the tour $\tau_{i,i+1}$ obtained by 1-shift is the same as the one obtained by 2-p-opt, and the expression for $\Delta' E_{i,i+1}$ may be derived by applying the equations derived for the 2-p-opt. By summing equations (39), (40) and by subtracting equation (41) we find

$$\begin{aligned} \Delta' E_{i,i+1} = & \left(\frac{1}{q_{i+1}} - 1\right) A_{i,2} + (q_{i+1} - 1)(B_{i,1} - B_{i,n-1}) \\ & + (q_i - 1)\left(A_{i+1,1} - A_{i+1,n-1}\right) + \left(\frac{1}{q_i} - 1\right) B_{i+1,2}. \end{aligned} \tag{45}$$

We will now focus on the more general case where $k \geq 2$ ($k \leq n-1$). We re-define the notions of *inside*, *outside* and of the contributions to the change in expected tour length adapting them for the 1-shift.

Definition 5 $inside_{i,j} = \{i+1, \ldots, j\}$, *that is, the 'inside' of the shifted section of* $\tau_{i,j}$

Definition 6 $outside_{i,j} =$ *all nodes* $\setminus (inside_{i,j} \cup \{i\})$

Definition 7 $\Delta' E_{i,j}\big|_{S \to T}$ *= the contribution to* $\Delta' E_{i,j}$ *due to the arcs connecting the nodes in* S *to the nodes in* T.

Definition 8 $\Delta' E_{i,j}\big|_{S\leftrightarrow T}$ *= the contribution to $\Delta' E_{i,j}$ due to the arcs connecting the nodes in S to the nodes in T and the nodes in T to the nodes in S.*

It is not difficult to verify that the weight on arcs between outside nodes and arcs between inside nodes does not change as a result of the shift. Therefore, the only contribution to $\Delta' E_{i,j}$ is given by the change in weight placed on arcs between $inside_{i,j} \cup \{i\}$ nodes and $outside_{i,j}$ nodes, and on arcs between node $\{i\}$ and $inside_{i,j}$ nodes, that is

$$\Delta' E_{i,j} = \Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}} + \Delta' E_{i,j}\big|_{\{i\}\leftrightarrow inside_{i,j}}. \tag{46}$$

In the following, we derive a recursive expression for each of the two contributions to $\Delta' E_{i,j}$. The contribution due to arcs between $inside_{i,j} \cup \{i\}$ and $outside_{i,j}$ for $\tau_{i,j}$ is the following

$$\begin{aligned}\Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}} &= \sum_{r=1}^{n-k-1}\Bigg[d(i-r,i)p_{i-r}p_i \prod_{i-r+1}^{i-1} q \left(\prod_{i+1}^{i+k} q - 1\right) \\ &\quad + d(i,i+k+r)p_i p_{i+k+r} \prod_{i+k+1}^{i+k+r-1} q \left(1 - \prod_{i+1}^{i+k} q\right)\Bigg] \\ &+ \sum_{t=1}^{k}\sum_{r=1}^{n-k-1}\Bigg[d(i-r,i+t)p_{i-r}p_{i+t} \prod_{i-r+1}^{i-1} q \prod_{i+1}^{i+t-1} q(1-q_i) \\ &\quad + d(i+k-t+1,i+k+r)p_{i+k-t+1}p_{i+k+r} \prod_{i+k-t+2}^{i+k} q \prod_{i+k+1}^{i+k+r-1} q(q_i-1)\Bigg]. \end{aligned} \tag{47}$$

To build the recursion, we must also consider the tour $\tau_{i,j-1}$. The contribution to $\Delta' E_{i,j-1}$ from arcs between $inside_{i,j-1} \cup \{i\}$ and $outside_{i,j-1}$ for $\tau_{i,j-1}$ is

$$\begin{aligned}\Delta' E_{i,j-1}\big|_{(inside_{i,j-1}\cup\{i\})\leftrightarrow outside_{i,j-1}} &= \sum_{r=1}^{n-k}\Bigg[d(i-r,i)p_{i-r}p_i \prod_{i-r+1}^{i-1} q \left(\prod_{i+1}^{i+k-1} q - 1\right) \\ &\quad + d(i,i+k-1+r)p_i p_{i+k-1+r} \prod_{i+k}^{i+k+r-2} q \left(1 - \prod_{i+1}^{i+k-1} q\right)\Bigg] \\ &+ \sum_{t=1}^{k-1}\sum_{r=1}^{n-k}\Bigg[d(i-r,i+t)p_{i-r}p_{i+t} \prod_{i-r+1}^{i-1} q \prod_{i+1}^{i+t-1} q(1-q_i) \\ &\quad + d(i+k-t,i+k+r-1)p_{i+k-t}p_{i+k+r-1} \prod_{i+k-t+1}^{i+k-1} q \prod_{i+k}^{i+k+r-2} q(q_i-1)\Bigg]. \end{aligned} \tag{48}$$

From the two expressions (47) and (48) it is possible to extract a recursive equation which relates $\Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}}$ to $\Delta' E_{i,j-1}\big|_{(inside_{i,j-1}\cup\{i\})\leftrightarrow outside_{i,j-1}}$

$$\Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}} = \Delta' E_{i,j-1}\big|_{(inside_{i,j-1}\cup\{i\})\leftrightarrow outside_{i,j-1}} + \delta, \tag{49}$$

which we will complete by expressing δ in terms of the matrices A and B. Observe that here, exactly like in the homogeneous PTSP [5], the difference between $\Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}}$ and $\Delta' E_{i,j-1}\big|_{(inside_{i,j-1}\cup\{i\})\leftrightarrow outside_{i,j-1}}$ will only involve arcs which are connected to nodes i and j, that is,

$$\begin{aligned}\delta = \;& \text{terms with } i \text{ and } j \text{ in } \Delta' E_{i,j}\big|_{(inside_{i,j}\cup\{i\})\leftrightarrow outside_{i,j}} \\ & - \text{terms with } i \text{ and } j \text{ in } \Delta' E_{i,j-1}\big|_{(inside_{i,j-1}\cup\{i\})\leftrightarrow outside_{i,j-1}}.\end{aligned} \tag{50}$$

So, by extracting the terms which contain the appropriate arcs from (47) and (48) and by expressing them in terms of the matrices A and B one obtains the following expression for δ

$$\begin{aligned}\delta = & \left(1 - \prod_{i+1}^{i+k} q\right)\left[\left(\prod_{i+1}^{i+k} q\right)^{-1} A_{i,k+1} - (B_{i,1} - B_{i,n-k})\right] \\ & + (q_i - 1)\left[(A_{j,1} - A_{j,n-k}) - q_i^{-1} B_{j,k+1}\right] \\ & - \left(1 - \prod_{i+1}^{i+k-1} q\right)\left[\left(\prod_{i+1}^{i+k-1} q\right)^{-1} A_{i,k} - (B_{i,1} - B_{i,n-k+1})\right] \\ & - (q_i - 1)\left[(B_{j,1} - B_{j,k}) - q_i^{-1} A_{j,n-k+1}\right],\end{aligned} \tag{51}$$

which completes the recursive expression (49).

Next we evaluate the contribution to $\Delta' E_{i,j}$ due to arcs between $\{i\}$ and *inside* (that is, the second term on the right side of equation (46). This contribution can be described by the following

$$\begin{aligned}\Delta' E_{i,j}\big|_{\{i\}\leftrightarrow inside_{i,j}} = & \left(\prod_{i+k+1}^{i+n-1} q - 1\right)\sum_{t=1}^{k}\left[d(i,i+t)p_i p_{i+t}\prod_{i+1}^{i+t-1} q\right. \\ & \left. - d(i+k-t+1,i)p_{i+k-t+1}p_i \prod_{i+k-t+2}^{i+k} q\right].\end{aligned} \tag{52}$$

Consider now the tour $\tau_{i,j-1}$. Then the contribution to $\Delta' E_{i,j-1}$ due to arcs between $\{i\}$ and $inside_{i,j-1}$ for $\tau_{i,j-1}$ is

$$\begin{aligned}\Delta' E_{i,j-1}\big|_{\{i\}\leftrightarrow inside_{i,j-1}} = & \left(\prod_{i+k}^{i+n-1} q - 1\right)\sum_{t=1}^{k-1}\left[d(i,i+t)p_i p_{i+t}\prod_{i+1}^{i+t-1} q\right. \\ & \left. - d(i+k-t,i)p_{i+k-t}p_i \prod_{i+k-t+1}^{i+k-1} q\right].\end{aligned} \tag{53}$$

From the two above expressions (52) and (53) it is possible to extract a recursive equation which relates $\Delta' E_{i,j}\big|_{\{i\}\leftrightarrow inside_{i,j}}$ to $\Delta' E_{i,j-1}\big|_{\{i\}\leftrightarrow inside_{i,j-1}}$:

$$\Delta' E_{i,j}\big|_{\{i\}\leftrightarrow inside_{i,j}} = \Delta' E_{i,j-1}\big|_{\{i\}\leftrightarrow inside_{i,j-1}} + \gamma. \tag{54}$$

Now, by subtracting (53) from (52) and by applying the definition of A (4), and B (5), we obtain the following expression for γ

$$\begin{aligned}\gamma = &\left(\prod_{i+k+1}^{i+n-1} q - 1\right)\left[A_{i,1} - A_{i,k+1} - \left(\prod_{i+k+1}^{i+n-1} q\right)^{-1} B_{i,n-k})\right] \\ &+ \left(1 - \prod_{i+k}^{i+n-1} q\right)\left[A_{i,1} - A_{i,k} - \left(\prod_{i+k}^{i+n-1} q\right)^{-1} B_{i,n-k+1}\right],\end{aligned} \tag{55}$$

which completes the recursive expression (54).

Let

$$P'_{i,j} = \prod_{i+1}^{j} q, \quad \overline{P'}_{i,j} = \prod_{j+1}^{i+n-1} q. \tag{56}$$

Based on equations (46), (49) and (54), we can write

$$\Delta' E_{i,j} = \Delta' E_{i,j-1} + \delta + \gamma,$$

which, given the expressions for δ (51) and γ (55) and the definitions for $P'_{i,j}$ and $\overline{P'}_{i,j}$ (56) becomes

$$\begin{aligned}\Delta' E_{i,j} = \Delta' E_{i,j-1} &+ (\overline{P'}_{i,j} - P'^{-1}_{i,j})(q_j A_{i,k} - A_{i,k+1}) \\ &+ (\overline{P'}^{-1}_{i,j} - P'_{i,j})(B_{i,n-k} - \frac{1}{q_j} B_{i,n-k+1}) \\ &+ (1 - \frac{1}{q_j}) P'_{i,j} B_{i,1} + (\frac{1}{q_i} - 1) B_{j,k+1} \\ &+ (1 - q_j)\overline{P'}_{i,j} A_{i,1} + (1 - \frac{1}{q_i}) A_{j,n-k+1} \\ &+ (q_i - 1)(A_{j,1} - A_{j,n-k}) \\ &+ (1 - q_i)(B_{j,1} - B_{j,k}).\end{aligned} \tag{57}$$

The 1-shift algorithm for the heterogeneous PTSP works like it does for the homogeneous PTSP, which is similar to the 2-p-opt. In the first phase of computation, $\Delta' E_{i,i+1}$ values for every i are computed by means of equation (45) and stored, while the two matrices of partial results A and B are accumulated. At the end of the first phase, the matrices P' and $\overline{P'}$ are computed. This phase requires $O(n^2)$ time, the same as with 2-p-opt. The second phase of the local search consists of computing $\Delta' E_{i,j}$ values recursively by means of equation (57). Like 2-p-opt, since each $\Delta' E_{i,j}$ in phase two is computed in $O(1)$ time, this phase, and thus the entire 1-shift checking sequence, may be performed in $O(n^2)$. With a straightforward implementation that does not utilize recursion, evaluating the 1-shift neighborhood would require $O(n^4)$ time instead of $O(n^2)$.

5 Conclusions

In this paper, we focused on the general PTSP problem where no assumption is made on the value of customer probabilities (heterogeneous PTSP). We have derived new expressions for the efficient computation of the expected cost of 2-p-opt and 1-shift local search moves. These derivations imply that it is possible to compute the cost evaluations of the entire neighborhood of a solution in $O(n^2)$ time, as in the homogeneous PTSP. Moreover, this result corrects the methods known in the literature and improves them by an $O(n)$ time factor.

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