Rotations from a New Perspective: Experiments, Modelling and Analogies Based on Angular Momentum as an Extensive Quantity

Alessandro Ascari, Fondazione Ducati, Bologna, Italy
Federico Corni, Università degli studi di Modena e Reggio Emilia, Modena, Italy
Tommaso Corridoni, SUPSI-DFA, Locarno, Switzerland
Michele D'Anna, Liceo Cantonale di Locarno, Locarno, Switzerland
Giovanni Savino, Fondazione Ducati, Bologna, Italy, Università degli Studi di Firenze, Firenze, Italy

Abstract

Concepts and mathematical instruments used in elementary mechanics are often perceived as abstract entities by students. We propose therefore an approach to the description of rotational mechanical processes based on a conceptualization of angular momentum grounded on image schemes and analogies from common everyday experience (Fuchs, 2007). In our approach angular momentum is described as a conserved extensive quantity, whose balance equation is either an instrument to foster clear mental images of rotational processes, or a solid base for algebraic development. Exploiting analogies, in this work we show how this way angular momentum can be stored in rotating bodies, how their moment of inertia represents their angular momentum capacities, and finally how a torque applied to a rotating body can be imagined as an angular momentum flow. We analyse from this point of view three experiments, using on-line data acquisition and dynamical modelling. The use of analogies allows indeed to develop dynamical models in a wide range of contexts, able to strengthen basic concepts and discuss phenomena in relation with the initial conditions and the parameter values.

Introduction

Educational research points out widespread learning problems in mechanics. We propose an approach based on a conceptual revision at the disciplinary level explaining how our usual perception of reality is grounded on image schemes, common among different aspects of human experience (Fuchs, 2007).

The usual description of mechanical processes starts from kinematics, although concepts and mathematical instruments are perceived this way as abstract entities by students.

Our approach is based on extensive mechanics quantities like momentum and angular momentum, whose balance equations, besides being a solid base for algebraic development, permit to develop clear mental images of processes. Coherently with this approach, we use analogy between models as a powerful way to understand mechanical systems and processes.

This contribution highlights the advantages of such an approach combined with technological tools as on-line data acquisition devices and dynamical modelling software.

In the next paragraph we show the conceptual framework, giving in the subsequent paragraphs an overview of three didactic experiences coming from the informal context of a science centre and the formal one of a school laboratory.

Angular momentum as stored quantity

Classical mechanics defines the angular momentum L stored in a rotating body starting from its moment of inertia I, referring to the rotation axis, and the angular velocity ω, according to:

$$L = I\,\omega \tag{1}$$

Further, the rate of change of angular momentum is related to the (total) torque τ acting on the body:

$$\frac{dL(t)}{dt} = \tau(t) \tag{2}$$

This balance law suggests that torque τ can be interpreted as the intensity of angular momentum flow, allowing the construction of a physical model by means of analogical reasoning, starting from simple experiences in hydraulic context.

Everyday life shows indeed that water can be stored in recipients, and its amount can be changed by a flow, i.e. an exchange with its surrounding. As a matter of fact, water can flow through holes or pipes connecting recipients with different levels of water.

In particular, the physical quantity *volume of water V,* stored in a recipient, can be expressed through the capacitive law:

$$V = C\ p$$

where C is the hydraulic capacity of the recipient and p the hydrostatic pressure at the bottom of the recipient itself, given by:

$$p = \rho\ g\ h$$

where h is the level of water, ρ its density and g the intensity of the gravitational field. In particular, for a recipient of constant section A we have:

$$V = A\,h \qquad\qquad C = \frac{A}{\rho\ g}$$

As the quantity *volume of water V*, also angular momentum can be thought by analogy as an extensive substance-like quantity which can be stored in a rotating body of moment of inertia I and angular velocity ω, so that also equation (1) represents a capacitive law, as equation (2) represents the corresponding balance equation. This means that a body changes its rotational state when an angular momentum flow, i.e. a torque, brings/takes away L to/from it. In the case of interacting bodies, therefore, the angular velocity difference is the quantity determining the direction of angular momentum's flow. In particular, considering only rotational phenomena, two bodies can exchange L only if they are rotating at different angular velocities (as water, in pure hydraulic processes, flows only when a pressure difference is given) (Fuchs, 2002; Fuchs, 2007).

This mental representation, obtained by analogy between models of extensive quantities behaviour, is the base of our approach to explain rotational mechanics.

We now show three didactical experiments to let the reader understand how the interpretation of rotational processes can be based on the point of view sketched above.

A. In the first example we present a rotational collision process, i.e. a short interaction between two rotating wheels. Angular momentum conservation will be discussed too.

B. The second example focuses instead on what happens during the interaction between two wheels exchanging angular momentum for a longer time.

C. In the last example, the angular momentum change in a body spinning and varying its moment of inertia is finally studied.

For each example we show at first a hydraulic model and then a quantitative analysis based on a dynamical model. According to the previous analogy between water volume and angular momentum, hydraulic models provide a powerful representation of mechanical processes, while dynamical models, designed with a software which allows to work on a graphical surface (Fuchs, 2002), translate the hydraulic model in a four-symbol language:

- the stock, representing the amount of a quantity which can be brought in or out via a flow;
- the flow, which can carry the same quantity from a stock to another one;
- the circle, which represents a parameter, as capacity, or a constant, as ρ or g;
- the arrow, which points out a relationship between stocks, flows and/or parameters.

The proposed models has been made using the software STELLA (www.iseesystems.com). Obviously, for this purpose several other computing instruments can be used: from an electronic spreadsheet to one of the wide variety of software products on the market.

2.A. Inelastic collision between two disks

Two coaxial Plexiglas disks are mounted on a low-friction axis (Fig.1). When the upper one is raised, the two disks rotate independently; when it is lowered they hit, bringing into contact each other via pieces of rubber (stuck on the upper surface of the lower disk).

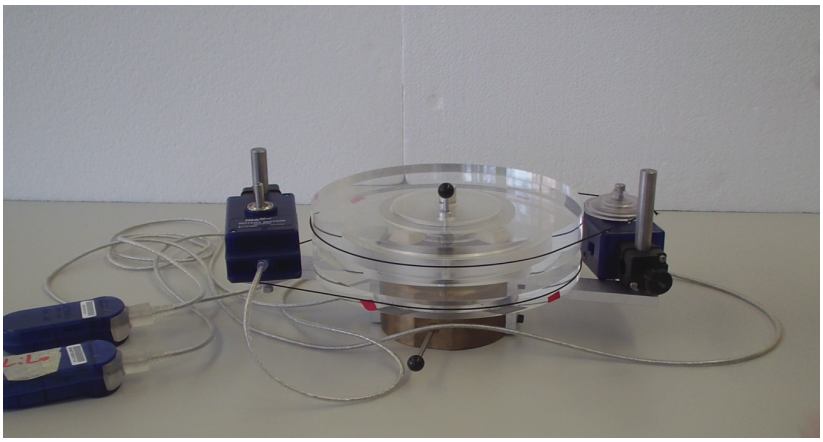

Figure 1. Experimental setup for inelastic collision between disks. Each disk is provided with a rotary motion sensor to measure angular velocity vs. time.

At the beginning of the experiment, the upper disk is raised, and both disks are put in rotation at different angular velocities. When the upper disk is abruptly lowered, it is slowed/accelerated from the lower one, which is instead accelerated/slowed, until the whole system of the two disks moves at the same angular velocity.

In the graph below (Fig. 2), the measured angular velocities of the disks are represented vs. time. The interaction time (~0,10 s) is short enough to allow us to neglect, during the inelastic collision, small friction effects.

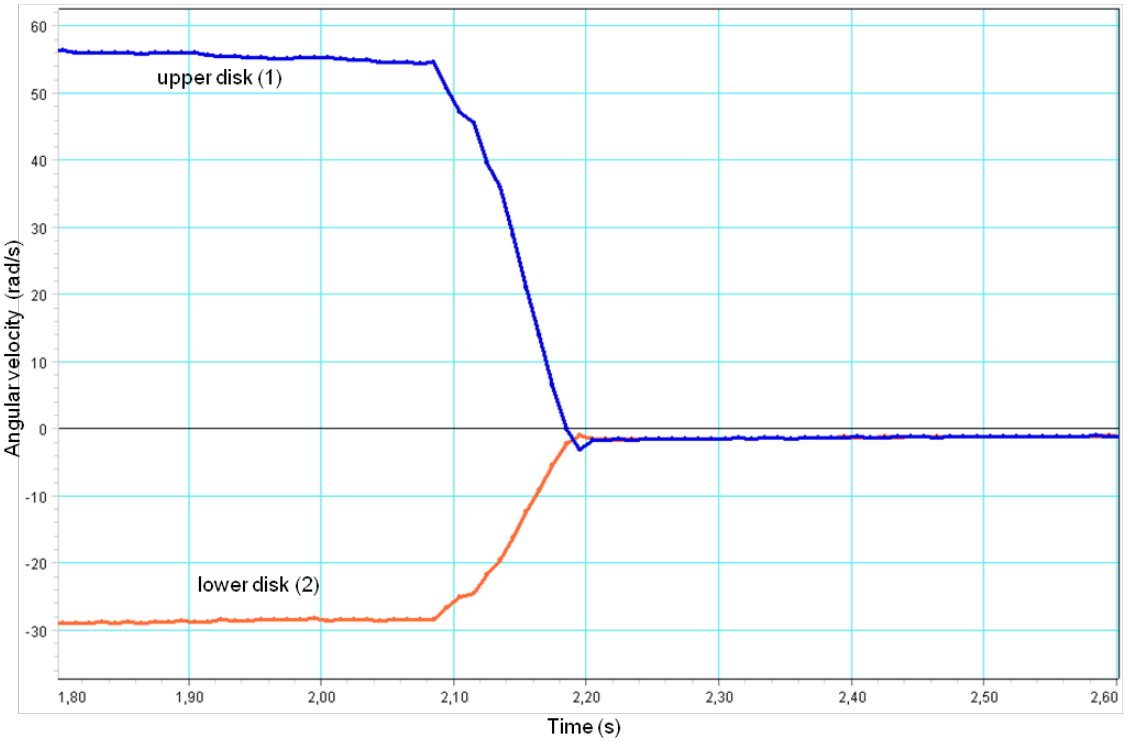

Figure 2. Experiment A: angular velocities measured vs. time. When the two disks are not in contact, their angular velocities change very slowly under the action of friction with the rotation axis. The interaction time (~0,10 s) is short enough to allow us to neglect this friction effect during the inelastic collision. The changes in the angular velocities during the collision are different because the two disks have different moments of inertia. The graph shows the angular velocity of the sensors: in order to obtain the angular velocity of the disks it is necessary to take into account the ratio between the radii of the pulley and the disk.

It is worthwhile to note at this point the analogy with inelastic collision between two carts moving with different speeds: as the first cart hits the second one, they move together, and their final common velocity, as well known, can be calculated using linear momentum conservation.

In the rotational collision, linear momentum is replaced by angular momentum and translational velocity by angular velocity, but the process can be described the same way.

When the two disks are separated, the angular momentum of the upper disk (L1) or that of the lower one (L2) change only by the action of the torque due to friction with the rotation axis, so that their angular velocities ($\omega 1$ and $\omega 2$ respectively) slowly decrease (Fig. 2). When the upper disk is lowered, as long as there is an angular velocity difference, an additional torque due to friction between the two disks appears.

Therefore there is a flow of angular momentum which intensity and direction depend on the chosen initial conditions. In Fig. 2, for instance, angular momentum flows from disk 1 to disk 2, raising its angular velocity $\omega 2$, and therefore its angular momentum L2 , being its moment of inertia constant. This process stops when both disks rotates at the same angular velocity, which, as well known, can be calculated using angular momentum conservation.

Following our approach, we now develop a hydraulic model of this first situation in order to reach a deeper and more general insight in dynamical systems time evolution, useful to obtain a model of rotational processes effective also in more complex situations, as those in examples B and C.

We replace the two disks with two water tanks connected by a pipe opened or closed with a tap (which modulates the interaction), each tank having a small hole in the bottom (representing the constant inner

friction of the support system). The two tanks have different base areas (i.e. different hydraulic capacities), so they contain different amount of water even if the water levels inside them are the same (Fig. 3).

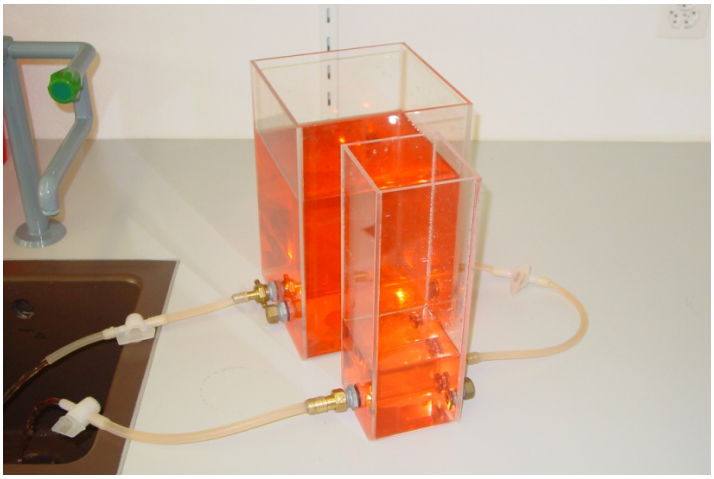

Figure 3. Experimental setup for hydraulic analogy: two communicating vessels are equipped with additional pipes in order to exchange water with the environment to. In each connection, a tap permits to modulate the intensity of the water flow.

The experiment is as follows: at the beginning the tap in the connecting pipe is closed and the two tanks are filled with different levels of water; only a small amount of water leaks through the holes in the bottom, lowering the levels of water. At a certain time, the tap in the connecting pipe is opened too, letting the water flow between the tanks. The water flow stops when the levels in the tanks are the same, i.e. when an equilibrium between pressures is reached.

The analogy is completed recognizing the following correspondences: water volume and angular momentum as extensive quantities subjected to balance equations; pressure differences and angular velocity differences as driving forces for hydraulic and rotational phenomena respectively; base areas of the tanks and moments of inertia of the disks as hydraulic and rotational mechanic capacities respectively.

In Fig. 4, this analogy is fostered by means of two dynamical models, one describing the two-disks system, the other the two tanks system. In both, stocks represent extensive quantities (angular momentum or water volume respectively) and blue thick arrows represent flows between stocks. The structure of the models is the same.

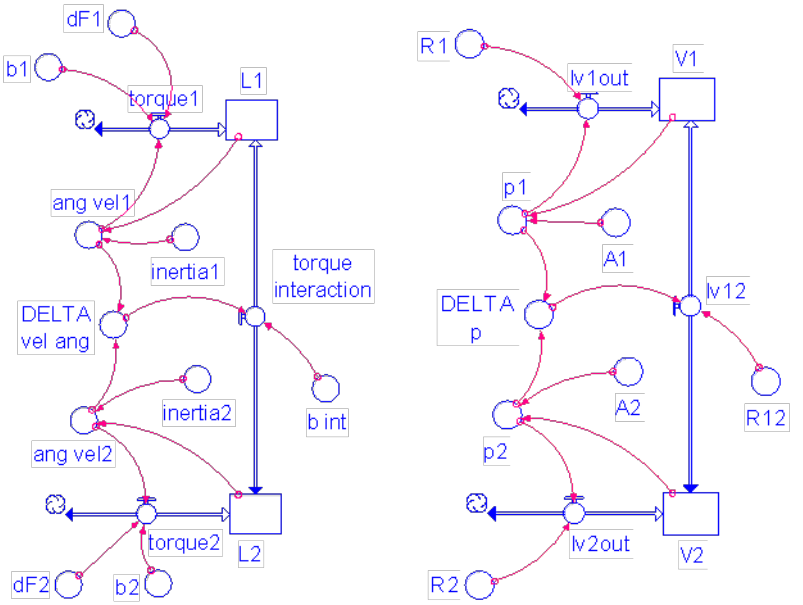

Figure 4. Dynamical models for the setup of experiment A (left) and its hydraulic analogue (right). For each disk/recipient, a tank for storage of angular momentum/water volume is introduced; thick lines represent the acting torques/water flows, i.e. the angular momentum/water volume flows. The parameter 'b int', related to the friction regulated by the clamp, has been imposed positive and constant only if there is an angular velocity difference.

The parameters of the dynamical models were determined independently with different experiments. We have interpreted the different interactions as constant flows over time.

Fig. 5 shows the angular velocities of the disks measured and calculated by the dynamical model, vs. time. The comparison is satisfactory, considering that the model reproduce the whole process, before, during and after the collision.

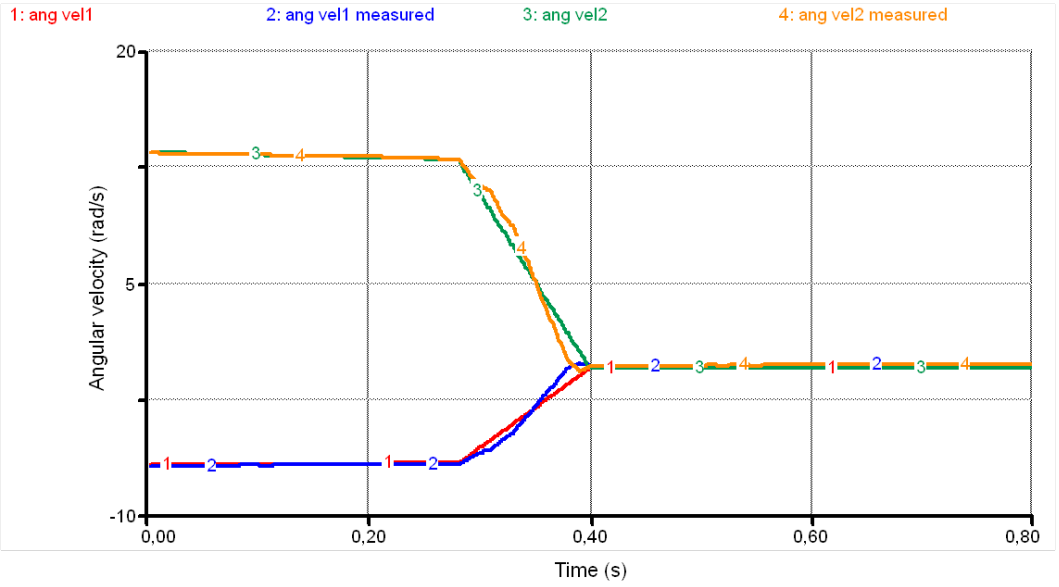

Figure 5. Experiment A: comparison of measured (2, 4) and calculated (1, 3) angular velocities. Even during collision, the model reproduces adequately the experimental data.

2.B. Magnetic interaction between two disks

In this second experiment the setup of the first is modified in order to control the interaction process between the two disks: they now interact by means of electromagnetic induction, whose length in time can be chosen a priori changing the distance between disks.

The apparatus used has (Fig. 6):

- an upper disk made of aluminium;
- a lower one made of plastic;
- the upper surface of the lower disk equipped with some magnets;
- the lower disk equipped with an adjustable (mechanical) friction, realized with a clamp.

Figure 6. Experimental setup for experiment B. Each disk is provided with a rotary motion sensor to measure angular velocity vs. time. The strength of the magnetic interaction can be changed varying the distance between disks.

At the beginning of the experiment, the lower disk is put in rotation, while the upper one is kept still by means of a stand, behaving as an electromagnetic brake. Then, the upper disk is released and it starts moving dragged by means of the electromagnetic inducted torque.

As in the previous experiment, both disks can possess angular momentum, but in this case only one is initially rotating. Furthermore, they experience different friction torques on their bearing systems. When the lower disk is rotating, its angular momentum decreases because of the torque produced by electromagnetic interaction between the magnets and the upper disk. Through this interaction, angular momentum is transferred from the lower to the upper disk, initially at rest, increasing its angular momentum, i.e. its angular velocity. The rate at which the angular momentum is transferred is given by the torque value, which is itself depending on the difference between the two angular velocities.

Regulating opportunely the three taps in the equipment shown in Fig. 3, the hydraulic system can behave analogously to the mechanical one for all different initial conditions. For instance, closing the tap in the connecting pipe partially, the intensity of the water flow can be reduced, extending therefore the interaction over a longer time.

Fig. 7 shows the dynamical model for the mechanical situation here considered: its structure is the same of that in Fig. 4, apart from the connection between torque interaction and torque friction for the upper disk, which permits to take into account the "grounding" of the upper disk while it is kept still by means of the stand (which obviously remains fixed to the Earth). The parameters describing the interactions were determined separately, with different experiments.

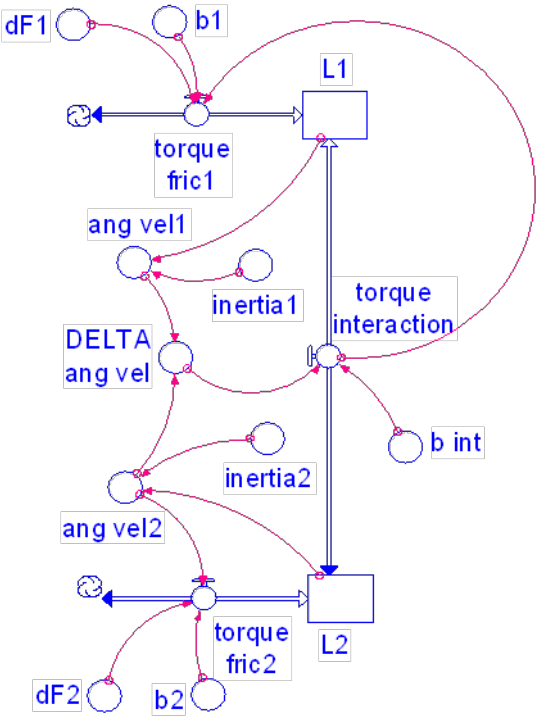

Figure 7. Dynamical model for the setup of experiment B. For each disk, a tank for storage of angular momentum is introduced; thick lines represent the acting torques, i.e. the angular momentum flows. The parameter defining the friction interaction can be measured in independent experiments for each disk.

Fig. 8 shows the angular velocities of the disks measured and calculated by the dynamical model, vs. time. The comparison is satisfactory. It is worth noting that with the chosen initial conditions, the sign of the difference between the angular velocities reverses (this means that the upper disk rotates longer than the lower one!).

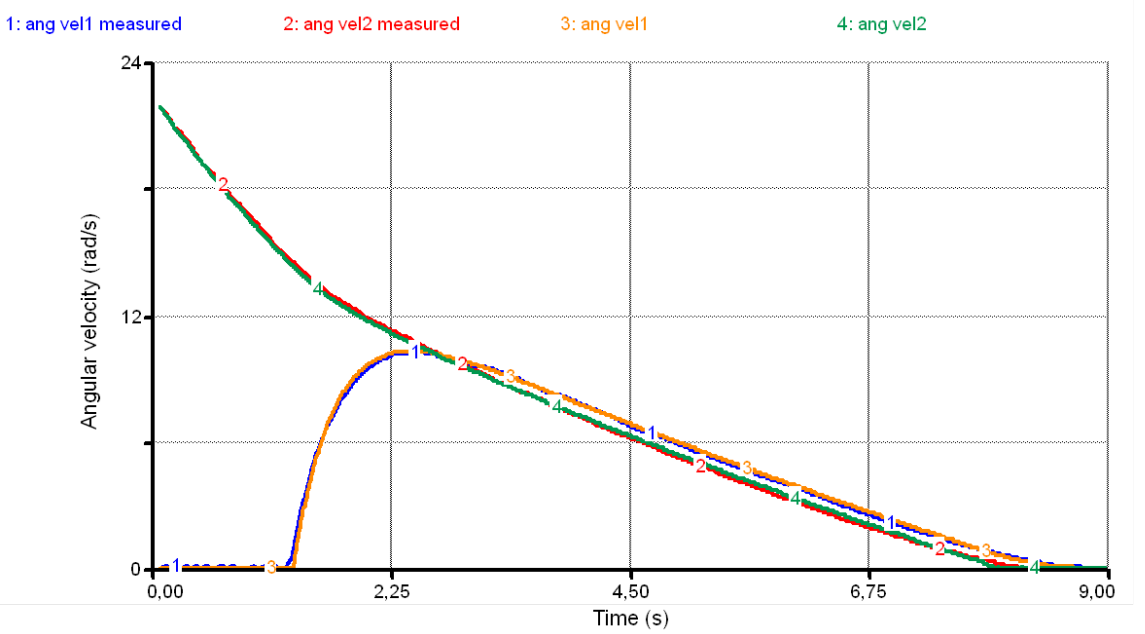

Figure 8. Experiment B: comparison of measured (1, 2) and calculated (3, 4) angular velocities. In the first part of the process, the upper disk is kept still.

2.C. Variable moment of inertia of a body

In experiences A and B the angular momentum has been transferred between systems (among disks or environment) by means of interactions (collision, friction, electromagnetic induction). The transfer process has been in both cases visualized by a hydraulic model and quantified by a dynamical model, in order to reproduce real measures vs. time assuming hypothesis *a priori*, as balance equations and angular momentum conservation.

In the experience C, another step toward the goal of our approach, i.e. to let students relate phenomena to a more general model of rotational dynamics, is done. While in experiences A and B the influence of interactions on the angular momentum levels, i.e. the angular velocities, has been studied taking constant the angular momentum capacities of the interacting systems, i.e. their moments of inertia, in experiment C, the dynamical behaviour of the system is driven essentially changing its moment of inertia.

The system chosen for the experiment is a beam rotating around a vertical axis passing through the midpoint of the beam itself. During rotation, its moment of inertia can increase or decrease by making two objects of equal masses move towards the rotation axis or the extremities of the beam (fig. 9). The two objects are moved using a lead screw actuated by an electric motor mounted on the beam itself. The first objects is a person sitting in security conditions on a motorcycle mock-up and the second one is a simple counterbalance mass. The apparatus belongs indeed to “Fisica in Moto” (Physics on Motorbikes) of Fondazione Ducati, an educational laboratory for high school students, developed by engineers and technicians of the motorcycle manufacturer Ducati Motor Holding S.p.A. (see www.fisicainmoto.it for further details).

Figure 9. Experimental setup for experiment C. The horizontal beam can rotate around the vertical symmetry axis. The two objects mounted on the beam have the same masses and have symmetric positions with respect to the rotation axis.

The red line in Fig. 10 represents the measured angular velocity. At the beginning of the experiment, the two objects are at the ends of the beam and the beam is still. During phase A in Fig. 10 a horizontal force (orthogonal to the beam) is applied at the extremity of the beam thus increasing the angular velocity of the system. In this phase the force intensity F is measured vs. time. At the beginning of phase B the force is removed and the system starts to slow down due to the friction with the rotating axis. At the beginning of phase C the two objects start moving along the beam, both in the direction of the rotation axis, thus decreasing the total moment of inertia of the system. By angular momentum conservation, when the moment of inertia decreases the angular velocity increases as stated by equation (1). The objects reach their final position at the centre of the beam at the end of phase C. During phase D the overall system angular velocity slows down because of the sole effect of friction with the rotation axis and finally comes to a stop.

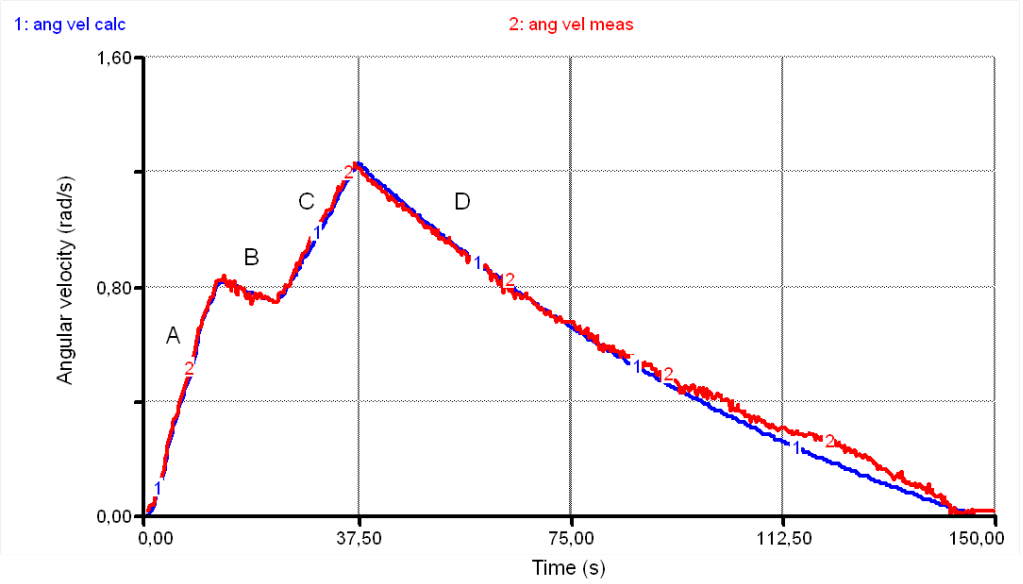

Figure 10. Experiment C: comparison of calculated (1) and measured (2) angular velocities. The model parameters are determined in the first two phases of the process (see text for further details).

The blue line in Fig. 10, refers to the results of the dynamical model represented in Fig. 11, accounting for all the phases of the considered process. Measures and dynamical model results match considering the torque needed to put in rotation the beam ('F measured' in Fig.11) and a friction. In particular, the friction ('torque friction' in Fig.11), consists of a constant part, related to the inner friction of the bearing system, and a variable one, related to the moving beam within the air, increasing as the square of the angular velocity.

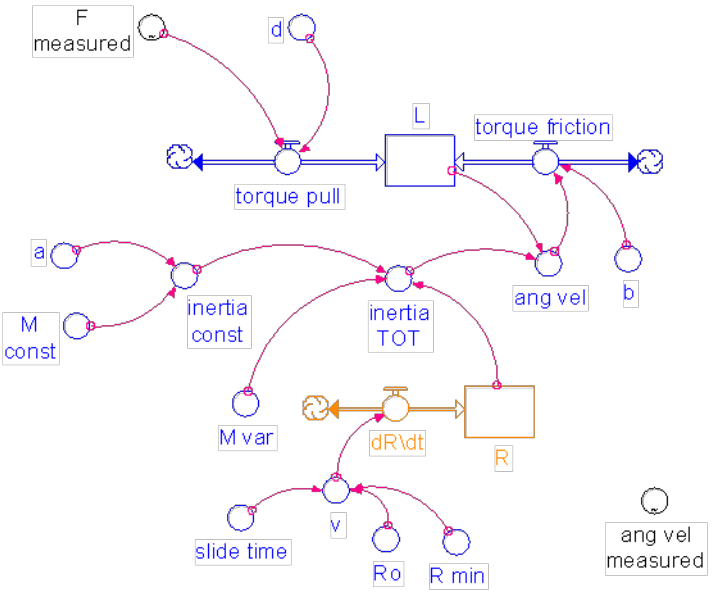

Figure 11. Dynamical model for the setup of experiment C. Only one tank, L, is needed to store the angular momentum of the whole system. The measured velocity v of the objects along the beam, allows to determine their distance R from the rotational axis vs. time, and therefore the moment of inertia as well. Thick lines represent the acting torques, i.e. the angular momentum flows.

3. Conclusions and outlook

We showed an approach to mechanical rotations based on angular momentum considered as a substance-like quantity. As a matter of fact we explained, by means of a friendly cognitive tool as hydraulic analogy, what happens during several rotational processes due to different kind of interactions, reproducing then with the help of dynamical modelling, the measured quantities involved, angular velocity, vs. time.

We showed three different real experiments. The first one, similar to a classical "linear" inelastic collision experiment, where essentially only the initial and final states are observed. Then, we showed a second more complex experiment, with a longer interaction time, permitting us to focus on the dynamics during the process. Finally, with the third experiment we considered a situation in which the moment of inertia was changed.

Our approach has been applied in both formal and informal contexts: while the first two experiments belong to a didactical physics laboratory of Liceo Cantonale of Locarno, the last one belongs to the science centre 'Fisica in Moto', a didactical laboratory of Ducati in Bologna.

Acknowledgements

We would like to express our gratitude to all the people who gave us the possibility to carry on this research.

We want to thank the headship of Liceo Cantonale of Locarno for hosting us and allowing us to use their devices. We thank also Mr. M. Martignoni (SAMB Bellinzona, CH) for the continuous and precise help in the realization of the devices used in experiments A and B.

We thank finally the management of Fondazione Ducati for the use of the device and the equipment used in experiment C.

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