On maximum number of minimal dominating sets in graphs

Fedor V. Fomin\(^{1,2}\)

Department of Informatics
University of Bergen
N-5020 Bergen, Norway

Fabrizio Grandoni\(^{3,4}\)

Dipartimento di Informatica
Università di Roma “La Sapienza”
Via Salaria 113, 00198 Roma, Italy

Artem V. Pyatkin\(^{5,6}\)

Department of Informatics
University of Bergen
N-5020 Bergen, Norway

Alexey A. Stepanov\(^7\)

Department of Informatics
University of Bergen
N-5020 Bergen, Norway
How many subgraphs of a given property can be in a graph on \( n \) vertices? This question is one of the basic questions in Graph Theory. For example, the number of 1-factors (perfect matchings) in a simple \( k \)-regular bipartite graph on \( 2n \) vertices is always between \( n!(k/n)^n \) and \( (k!)^{n/k} \). (The first inequality was known as van der Waerden Conjecture [8] and was proved in 1980 by Egorychev [3] and the second is due to Bregman [1].) Another example is the famous Moon and Moser [7] theorem stating that every graph on \( n \) vertices has at most \( 3^{n/3} \) maximal independent sets. Such combinatorial bounds are of interests not only on their own but also because they are used for algorithm design as well. Lawler [6] used Moon-Moser bound on the number of maximal independent sets to construct \( \mathcal{O}((1 + \sqrt[3]{3})^n) \) time graph coloring algorithm which was the fastest coloring algorithm for 25 years. Recently Byskov and Eppstein [2] obtain \( \mathcal{O}(2.1020^n) \) time coloring algorithm which is also based on a combinatorial bound \( 1.7724^n \) on the number of maximal bipartite subgraphs in a graph.

Let \( G = (V, E) \) be a graph. A set \( D \subseteq V \) is called a dominating set for \( G \) if every vertex of \( G \) is either in \( D \), or adjacent to some node in \( D \). A dominating set is minimal if all its proper subsets are not dominating. We define \( \text{DOM}(G) \) to be the number of minimal dominating sets in a graph \( G \). The Minimum Dominating Set problem (MDS) asks to find a dominating set of minimum cardinality. Despite of importance of minimum dominating set problem on which hundreds of papers have been written (see e.g. the surveys [4,5] by Haynes et al.), nothing better the trivial \( \mathcal{O}(2^n/\sqrt{n}) \) bound was known for \( \text{DOM}(G) \). In this paper we prove the following

**Theorem.** For every graph \( G \) on \( n \) vertices \( \text{DOM}(G) \leq 1.7697^n \).

**Proof.** First we reduce MDS to the Minimum Set Cover problem (MSC). In this problem we are given a hypergraph \( H = (U, S) \) with a vertex set \( U \) and an edge set \( S \) of (non-empty) subsets of \( U \). The aim is to determine the minimum
A covering is minimal if it contains no smaller covering. We denote by $\text{COV}(H)$ the number of minimal coverings in $H = (U, S)$.

The problem of finding $\text{DOM}(G)$ can be naturally reduced to finding $\text{COV}(H)$ by imposing $U = V$ and $S = \{N[v] \mid v \in V\}$. Note that $N[v] = \{v\} \cup \{u \mid uv \in E\}$ is the set of nodes dominated by $v$. Thus $D$ is a dominating set of $G$ if and only if $\{N[v] \mid v \in D\}$ is a set cover of $H = (U, S)$. So, each minimal set cover of $H$ corresponds to a minimal dominating set of $G$.

Consider now an arbitrary example of the MSC problem with a hypergraph $H = (U, S)$. Denote by $s_i$ the number of edges of cardinality $i$ for $i = 1, 2, 3$ and by $s_4$ the number of edges of cardinality at least 4 in $S$. Let $k = |U| + \sum_{i=1}^{4} \varepsilon_i s_i$ be the size of the MSC problem $(U, S)$. Here $\varepsilon_1 = 2.9645, \varepsilon_2 = 3.5218, \varepsilon_3 = 3.9279$, and $\varepsilon_4 = 4.1401$ are the size coefficients for the edges of cardinalities 1, 2, 3, and at least 4 respectively. Let $\text{COV}(k)$ be the maximum value of $\text{COV}(H)$ among all MSC problems of size $k$. We will prove that $\text{COV}(k) \leq \alpha^k$, where $\alpha \approx 1.11744562 < 1.1175$.

We use induction on $k$. Clearly, $\text{COV}(0) = 1$. Suppose that $\text{COV}(l) \leq \alpha^l$ for every $l < k$. Let $S$ be a set of subsets of $U$ such that the MSC problem $(U, S)$ is of size $k$. Let $d_2 = \min\{\varepsilon_1, \varepsilon_2 - \varepsilon_1\}, d_3 = \min\{d_2, \varepsilon_3 - \varepsilon_2\}$, and $d_4 = \min\{d_3, \varepsilon_3 - \varepsilon_2\}$. We consider different cases.

**Case 0.** There is a vertex $u \in U$ of degree 1. Since $u$ must be covered by the only set $S$ containing it, we may remove $u$ and $S$ along with all vertices from $S$ and reduce the size of the instance.

**Case 1.** $H$ has a vertex $u$ belonging to loops (edges of cardinality 1) only. Let $S_1 = S_2 = \ldots = S_r = \{u\}$, where $r \geq 2$ be all the edges containing $u$. Then every minimal covering should contain exactly one of them. Thus

$$\text{COV}(H) \leq r \cdot \text{COV}(k - r\varepsilon_1 - 1) \leq r\alpha^{k-r\varepsilon_1-1}.$$

**Case 2.** $H$ contains an edge of cardinality $r \geq 5$. Let $S = \{u_1, u_2, \ldots, u_r\}$ be such an edge. The number of minimal set covers that do not contain $S$ is at most $\text{COV}(U, S \setminus S)$ and the number of minimal set covers containing $S$ is at most $\text{COV}(U \setminus \{u_1, u_2, \ldots, u_r\}, S')$. Here $S'$ consists of all nonempty subsets $S' \setminus \{u_1, u_2, \ldots, u_r\}$ where $S' \in S$. Therefore

$$\text{COV}(H) \leq \text{COV}(k - \varepsilon_4) + \text{COV}(k - 5 - \varepsilon_4) \leq \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4}.$$

**Case 3.** $H$ contains an edge of cardinality 4. Let $S = \{u_1, u_2, u_3, u_4\}$ be such an edge. Again, $\text{COV}(H) \leq \text{COV}(U, S \setminus S) + \text{COV}(U \setminus \{u_1, u_2, u_3, u_4\}, S')$. 

Cardinality of a subset $S^* \subseteq S$ which covers $U$, i.e. such that

$$\bigcup_{S \in S^*} S = U.$$
But now removal of every vertex $u_1, u_2, u_3, u_4$ from $U$ reduces the size of the problem by at least $d_4 + 1$, since all edges contain at most 4 vertices and the minimum degree is 2. Thus

$$\text{COV}(H) \leq \text{COV}(k - \varepsilon_4) + \text{COV}(k - \varepsilon_4 - 4(d_4 + 1)) \leq \alpha^{k - \varepsilon_4} + \alpha^{k - 4 - \varepsilon_4 - 4d_4}.$$ 

We omit the detailed analysis for the next four cases since it is similar to the previous cases but more tedious.

**Case 4.** There is $u \in U$ of degree 2. Then

$$\text{COV}(H) \leq \min\{\alpha^{k - 1 - \varepsilon_1 - \varepsilon_2} + \alpha^{k - 2 - \varepsilon_1 - \varepsilon_2 - 3d_2}, 2\alpha^{k - 2 - 2\varepsilon_2}, 2\alpha^{k - 2 - 2\varepsilon_2 - d_3} + \alpha^{k - 3 - 2\varepsilon_2 - 2d_2}\}.$$ 

**Case 5.** $H$ contains an edge of cardinality 3. Then

$$\text{COV}(H) \leq \alpha^{k - \varepsilon_3} + \alpha^{k - 3 - \varepsilon_3 - 6d_3}.$$ 

**Case 6.** There are $S, S' \in S$ such that $S' \subset S$. In this case

$$\text{COV}(H) \leq \min\{\alpha^{k - \varepsilon_2} + \alpha^{k - 2 - \varepsilon_1 - \varepsilon_2 - 3d_2}, \alpha^{k - \varepsilon_2} + \alpha^{k - 2 - 2\varepsilon_2 - 2d_2}\}.$$ 

**Case 7.** There is $u \in U$ of degree 3. Then

$$\text{COV}(H) \leq 3\alpha^{k - 2 - 2\varepsilon_2 - 2d_2} + 3\alpha^{k - 3 - 6\varepsilon_2} + \alpha^{k - 4 - 6\varepsilon_2}.$$ 

**Case 8.** $H$ does not satisfy any of the conditions from Cases 1–7. In this case $H$ is an ordinary graph of minimum degree at least 4. Let $S = \{u, v\}$ be an edge of $H$. Denote by $S_u$ and $S_v$ the sets of all other edges containing $u$ and $v$ respectively. Clearly, $|S_u| \geq 3$ and $|S_v| \geq 3$. By the definition of minimality, if $S^*$ is a minimal cover containing $S$ then $S^* \cap S_u = \emptyset$ or $S^* \cap S_v = \emptyset$. So, we have at most $\text{COV}(k - \varepsilon_2)$ minimal covers that do not contain $S$ and at most $2 \cdot \text{COV}(k - 2 - 4\varepsilon_2 - 3d_2)$ covers containing $S$. Then

$$\text{COV}(H) \leq \alpha^{k - \varepsilon_2} + 2\alpha^{k - 2 - 4\varepsilon_2 - 3d_2}.$$ 

Summarizing, we have the following inequalities:
\[\alpha^k \geq \max \left\{ \begin{array}{l}
 r\alpha^{k-r\varepsilon_1-1}, \ r \geq 2 \\
 \alpha^{k-\varepsilon_4} + \alpha^{k-5-\varepsilon_4} \\
 \alpha^{k-\varepsilon_4} + \alpha^{k-4-\varepsilon_4-4d_4} \\
 \alpha^{k-1-\varepsilon_1-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-d_3} \\
 2\alpha^{k-2-2\varepsilon_2} \\
 2\alpha^{k-2-2\varepsilon_2-d_3} + \alpha^{k-3-2\varepsilon_2-2d_3} \\
 \alpha^{k-\varepsilon_3} + \alpha^{k-3-\varepsilon_3-6d_3} \\
 \alpha^{k-\varepsilon_2} + \alpha^{k-2-\varepsilon_1-\varepsilon_2-3d_2} \\
 \alpha^{k-\varepsilon_2} + \alpha^{k-2-2\varepsilon_2-2d_2} \\
 3\alpha^{k-2-3\varepsilon_2-2d_2} + 3\alpha^{k-3-6\varepsilon_2} + \alpha^{k-4-6\varepsilon_2} \\
 \alpha^{k-\varepsilon_2} + 2\alpha^{k-2-4\varepsilon_2-3d_2}
\end{array} \right\} \]

It could be checked (by using computer) that all of them hold for given \(\alpha\) and \(\varepsilon_i, \ i = 1, 2, 3, 4\). Therefore, \(\text{COV}(k) \leq \alpha^k < 1.1175^k\). But for any graph \(G\) on \(n\) vertices the corresponding instance of \(\text{MSC}\) problem has size at most \(|U| + \varepsilon_4|S| = (1 + \varepsilon_4)n\). Thus \(\text{DOM}(G) \leq \text{COV}((1 + \varepsilon_4)n) < 1.1175^{5.1401n} \leq 1.7697^n\), finishing the proof.

Note finally that the best known lower bound for \(\text{DOM}(G)\) is \(15^{n/6} \approx 1.5704^n\) (consider \(n/6\) disjoint copies of the octahedron). This example was found by Dieter Kratsch.

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References


