Some current researches in progress on the meaning of mathematical objects and about relations between semiotics and didactic at the NRD of Bologna

Bruno D'Amore – Martha Isabel Fandiño Pinilla – George Santi – Silvia Sbaragli
NRD – Department of Mathematics – University of Bologna – Italy[1]
www.dm.unibo.it/rsddm

Abstract. *Between the years 1995 and 2010 we developed some researches within the Research Group NRD Bologna (Italy) which led to investigate various salient and relevant aspects of Mathematics Education; on these issues the NRD has published numerous books and articles, participating in international conferences. We present here some research questions and some results.*

1. The phenomenon of change of the meaning of mathematical objects due to the passage between their different representations: how other disciplines can be useful to the analysis

1.1. Background

In D'Amore and Fandiño Pinilla (2007a, b), we reported and discussed, exclusively from a structural semiotic point of view, episodes taken from classroom situations in which students are mathematics teachers in their initial training, engaged in facing representations problems. Some examples of the phenomenon have been given orally in Rhodes, on April 13th 2006, during a general conference **(**How the treatment or conversion changes the sense of mathematical objects) at the 5th MEDCONF2007 (Mediterranean Conference on Mathematics Education), 13-15 April 2007, Rhodes, Greece (D'Amore, 2007).
The task consisted in this: working in small groups the trainee teachers received a text written in natural language; such texts had to be transformed into algebraic language. Once they had come to the algebraic formulation, this was explained by the group and collectively discussed. Our duty as university teachers was to suggest the further transformation of the obtained algebraic expressions into other algebraic *expressions*, to face collective discussions on their meaning.
We present three examples below.

Example 1
[We omit the original linguistic formulation which, in this case, is not relevant];
The final algebraic formulation proposed by group 1 is: $x^2+y^2+2xy-1=0$, which in natural language is interpreted as follows: «A circumference» [the interpretation error is evident, but we decide to

[1] This work is part of the PRIN research program (Programmi di ricerca scientifica di rilevante interesse nazionale / National Scientific Research Programs of National Relevant Interest): *Teaching mathematics: conceptions, good practices and teacher training*, 2008, Protocol # 2008PBBWNT, Bologna Local Unit (NRD, Department of Mathematics): *Training of Mathematics Teachers*.

pass over]; we carry out the transformation which leads to: $x+y=\frac{1}{x+y}$ that after a few attempts is interpreted as «A sum that has the same value of its reciprocal»;
question: «But $x+y=\frac{1}{x+y}$ is it or not the "circumference" we started with?»;
student A: «Absolutely no, a circumference must have x^2+y^2»;
student B: «If we simplify, yes».
One can ask whether or not it is the transformation that gives a *sense*: from the episode it seems that if one would perform the inverse passages, then one would return to a "circumference". But it could also instead be that the meanings are attributed to the specific representations, without links between them, as if the transformation that makes sense for the teacher it does not make sense for the person who performs it.

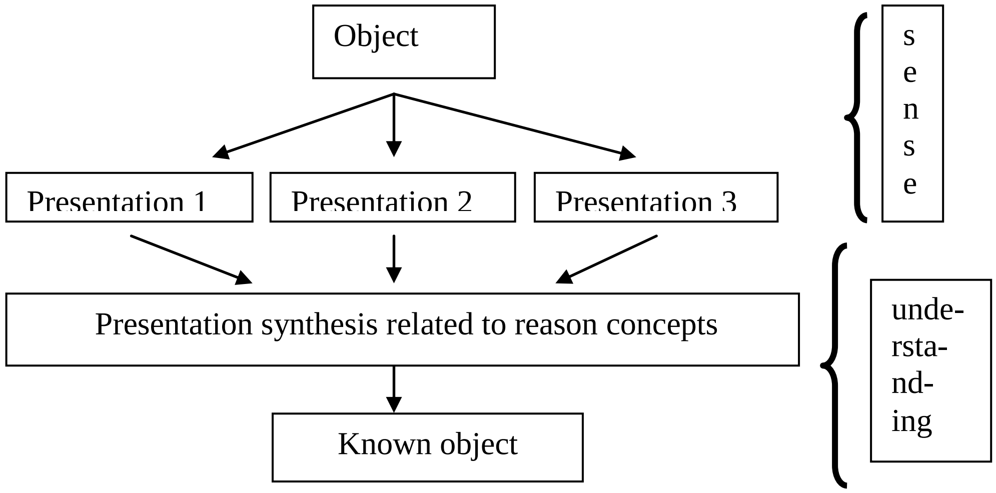
Example 2
The text written in natural language requires the algebraic writing of the sum of three consecutive natural numbers and the proposal of group II is: (n-1)+n+(n+1) [obviously the doubt remains in the case of n=0, but we decide to pass over]; we carry out the transformation that leads to the following writing: 3n that is interpreted as: «The triple of a natural number»;
question: «But 3n can be thought as the sum of three consecutive natural numbers?»;
student C: «No, *like this* no, *like this* it is the sum of three equal numbers, that is n».

Example 3
We consider the sum of the first 100 natural positive numbers: 1+2+…+99+100; we perform Gauss classical transformation; 101×50; this representation is recognized as the solution of the problem but not as the representation of the starting object; the presence of the multiplication sign compels all the students to look for a sense in mathematical objects in which the "multiplication" term (or similar terms) appears;
question: «But 101×50 is it or not the sum of the first 100 positive natural numbers?»;
student D: «That one, is not a sum, that is a multiplication; it corresponds to the sum, but it is not the sum».

In these episodes we witness a constant change of meaning during the transformations: each new representation has a specific meaning of its own not referable to the one of the starting representations, even if the passage from the first to the second ones has been performed in an evident and shared manner.

1. 2. The causes of the changes of meaning

What are the causes of the changes of meaning, what origin do they have?
We can start from this diagram that we appreciate a lot because of its attempt to put in the right place the ideas of *sense* and *understanding* (Radford, 2004a).



The process of meanings endowment moves at the same time within various semiotic systems, simultaneously activated; we are not dealing with a pure classical dichotomy: treatment/conversion leaves the meaning prisoner of the internal semiotic structure, but with something much more complex. Ideally, from a structural point of view, the meaning should come from within the semiotic system we are immersed in. Therefore, in *Example 2*, the pure passage from $(n-1)+n+(n+1)$ to $3n$ should enter the category: treatment semiotic transformation. But what happens in the classroom practice, and not only with novices in algebra, is different. There is a whole path to cover, starting from single specific meanings culturally endowed to the signs of the algebraic language ($3n$ is the triple of something; 101×50 is a product, not a sum). Thus, there are sources of meanings relative to the algebraic language that anchor to meanings culturally constructed, previously in time; such meanings often have to do with the arithmetic language. From an, so to speak, "external" point of view, we can trace back to seeing the different algebraic writings as equally significant since they are obtainable through semiotic treatment, but from inside this picture is almost impossible, bound as it is to the culture constructed by the individual in time. In other words, we can say that students (not only novices) turn out bridled to sources of meaning that cannot be simply governed by the syntax of the algebraic language. Each passage gives rise to forms or symbols to which a specific meaning is recognised because of the cultural processes THROUGH which it has been introduced.

In Luis Radford's mathematical knowledge is seen as the product of a reflexive cognitive mediated praxis. «Knowledge as cognitive praxis (*praxis cogitans*) underlines the fact that what we know and the way we come to know it are underpinned by ontological positions and by cultural processes of meaning production that give form to a certain way of rationality within which certain types of questions and problems are posed. The *reflexive* nature of knowledge must be understood in Ilyenkov's sense, that is, as a distinctive component that makes cognition an intellectual reflexion of the external world in accordance with the forms of individuals' activity (Ilyenkov, 1977, page 252). The mediated nature of knowledge refers to the role played by tools and signs as means of knowledge objectification and as instruments that allow us to bring to a conclusion the cognitive *praxis*» (Radford, 2004b, page 17).

On the other hand, «the object of knowledge is not filtered only by our senses, as it appears in Kant, but overall by the cultural modes of signification (...). (...) the object of knowledge is filtered by the technology of the semiotic activity. (...) knowledge is culturally mediated» (Radford, 2004b, page 20). «(…) These terms are the semiotic means of objectification. Thanks to these means, the general object that always remains directly inaccessible starts to take form: it starts to become an "object of consciousness" for the pupils. Although general, these objects however remain *contextual*» (Radford, 2004b, page 23).

The approach to the object and its appropriation on the part of the individual who learns, are the result of personal intentions with which individuals express themselves through experiences that see the objects used in suitable contexts: «Intentions occur in contextual experiences that Husserl called *noesis*. The conceptual content of such experiences he termed *noema*. Thus, noema corresponds to the way objects are grasped and become known by the individuals while noesis relates to the modes of cultural categorical experiences accounting for the way objects become attended and disclosed (Husserl, 1931)» (Radford, 2002, page. 82).
In the cases we presented above, and in mathematics in general, it is clear that the objects are attended from the first moment in their formal expression, in our case in the algebraic language; the individual learns to formally handle these signs, but what happens to the initial mathematical object? What happens to the initial meanings? We suppose that these meanings are tightly bound to the arithmetic experience of the pupil and overall to the way in which such an experience becomes objective through its objective transposition into ordinary language. Deep understanding of algebraic or, in general, formal manipulation, holds a prominent position.
Through an interesting comparison, Radford expresses himself on this point as follows: «While Russell (1976, page 218) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention, a noematic change: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl, 1961, page 79), which led him talk about signs having a *game signification* (...)» (Radford, 2002, page 88).
After having shown the broad and complex significance of the phenomenon, we must refer to other disciplines in order to understand better and better the issue of the different meanings of algebraic expressions, that is, in order to give a significant contribution to this aspect of mathematics education.

1.3. Analysis of the phenomenon thanks to theories "external" of mathematics education

We believe that some theories "external" of mathematics education can have, and in fact they already have, a strong influence on the analyses of various phenomena, like the ones described here, therefore giving a contribution to changing the theoretical frame of our discipline in its future research developments.

Philosophy. In section 1.2, we have seen how philosophy (Husserl's phenomenology) can have remarkable contribution and we will not repeat ourselves.
Learning is taking consciousness of a general object in accordance with the modes of rationality of the culture one belongs to.
More importantly we must face here the issue of the philosophical dilemma on concept and object, and even more the problem of the need of a previous choice between realist and pragmatist positions (D'Amore, Fandiño Pinilla, 2001; D'Amore, 2003; D'Amore, 2007).
In **realist theories** the meaning is a «conventional relationship between signs and ideal or concrete entities that exist independently of linguistic signs; they therefore suppose a conceptual realism» (Godino, Batanero, 1994). As Kutschera (1979) already claimed: «According to this conception the meaning of a linguistic expression does not depend on its use in concrete situations, but it happens that the use holds on meaning, since a clear distinction between pragmatics and semantics is possible».
In the realist semantics that it derives, we attribute to linguistic expressions purely semantic functions; the meaning of a proper name (as: 'Bertrand Russell') is the object that such proper name indicates (in such a case: Bertrand Russell); the individual statements (as: 'A is a river') express facts that describe reality (in such a case; A is the name of a river); the binary predicates (as: 'A reads B') designate attributes, those indicated by the phrase that expresses them (in this case: person

A reads thing B). Therefore every linguistic expression is an attribute of certain entities: the nominal relationship that derives is the only semantic function of expressions.
We recognise here the bases of Frege's, Carnap's and Wittgenstein's (*Tractatus)* positions.
A consequence of this position is the acknowledgement of a "scientific" observation (at the same time therefore, empiric and subjective or inter-subjective) as it could be, at a first level, a statement and predicate logic.
From the point of view we are mostly interested in, if we apply to Mathematics the ontological assumption of realist semantics, we necessarily draw a platonic picture of mathematical objects: notions, structures, etc. have a real existence that does not depend on human being, as they belong to an ideal domain; "to know" from a mathematical point of view means "to discover" in such domain entities and relationships between them. It is also obvious that such picture implies an absolutism of mathematical knowledge, since it is thought as a system of external certain truths that cannot be modified by human experience because they precede or, at least, are extraneous and independent from it.
Akin positions, although with different nuances, were sustained by Frege, Russell, Cantor, Bernays, Goedel,…; they also encountered violent criticisms [Wittgensteins' *Conventionalism* and Lakatos' *quasi-empirism* : see Ernest (1991) and Speranza (1997)].

In **pragmatic theories** linguistic expressions have different meanings according to the context in which they are used and therefore any scientific observation is impossible, since the only possible analysis is a "personal" and subjective one, anyway circumstantial and not generalizable. We cannot but analyse the different "uses": the set of "uses" in fact determines the meaning of objects.
We recognize here Wittgenstein's positions of the *Philosophical Investigations*, when he admits that the significance of a word depends on its function in a "linguistic game", since in such game it has a way of 'use' and a concrete purpose for which it has been precisely used: therefore the word does not have a meaning *per se*, but nevertheless, it can be meaningful.
Mathematical objects are therefore symbols of cultural units that emerge from a system of uses that characterise human pragmatics (or at least of individuals' homogeneous groups) and that continuously modify in time, also according to needs. In fact, mathematical objects and the meaning of such objects depend on the problems that we face in Mathematics and on their solution processes.

	"REALIST" THEORIES	**"PRAGMATIC" THEORIES**
meaning	conventional relationship between signs and concrete or ideal entities independent of linguistic signs	depends on the context and use
semantics Vs pragmatics	clear distinction	no distinction or faded distinction
objectivity or intersubjectivity	complete	missing or questionable
semantics	linguistic expressions have purely semantic functions	linguistic expressions and words have "personal" meanings, are meaningful in suitable contexts, but they don't have absolute meanings *per se*
analysis	possible and licit: logic for example	only a "personal" or subjective analysis is possible, not generalizable, not absolute
consequent	platonic conception of mathematical objects	problematic conception of mathematical objects

epistemological picture		
to know	to discover	to use in suitable contexts.
knowledge	is an absolute	is relative to circumstance and specific use
examples	Wittgenstein in *Tractatus*, Frege, Carnap [Russell, Cantor, Bernays, Gödel]	Wittgenstein in *Philosophical Invesigations* [Lakatos]

It is obvious and it would be easy to prove with philosophical examples, that the two fields are not fully complementary and clearly separated even if, for reasons of clarity, we preferred giving this "strong" impression.
With regard to the philosophical bases of Mathematics Education, we have decided to stay in the pragmatic domain that seems much closer to the reality of the empiric process of Mathematics teaching/learning. It seems that each specification that appears in the right column, cell by cell, is part of the same process and of its explicitation. It seems that focusing on didactical activity (and therefore research), on learning, and consequently on the epistemology of the domain that has the student as a protagonist, we are obliged to interpret each step of knowledge construction as responding to the *language game*, therefore admitting that the semantics blurs in pragmatics.

Sociology. In D'Amore (2005) and D'Amore and Godino (2007), we show how the results of the analyses relative to the behaviours of individuals engaged in an activity of conceptual learning of mathematical objects, their transformations of the descriptions of such objects from ordinary language to formal language, the manipulations of such formalizations can be framed within a sociological interpretation key: the learning environment is framed within a sociological interpretation key and the individuals' behaviours are interpreted through the notion of "practice" and its "meta-practice" evolution. Essentially the individuals shift from a shared practice, recognized as characteristic of the social group they belong to, to a meta-practice that modifies such a characteristic; the interpretative behaviour therefore ceases to be global and social and becomes local and personal; the notions that come into play in such interpretations are specific of the circumstance and not of the situation in its entirety.
We pass over this point, referring back to the quoted texts.

Anthropology. In D'Amore and Godino (2006, 2007) we go into strongly anthropological details in order to explain the nature of the choices of the individual who learns mathematics. In such articles we highlight how «Having obliged the researcher to point all his attention to the activities of human beings who have to do with mathematics (not only solving problems, but also communicating mathematics) is one of the merits of the anthropological point of view, inspiring other points of view, amongst which the one that today we call "anthropological" in the proper sense: the ATD, anthropological theory of didactics (of mathematics) (Chevallard, 1999; page 221). Why this adjective "anthropological"? It is not an exclusiveness of the approach created by Chevallard in 80s, as he himself declares (Chevallard, 1999), but an "effect of the language" (page 222); it distinguishes the theory, identifies it, but it is not peculiar to such theory in a univocal way» (D'Amore, Godino, 2006, page 15). The ATD is almost exclusively centred on the institutional dimension of mathematical knowledge, as a development of the research program started with fundamental didactics. The crucial point is that «ATD places the *mathematical* activity, and therefore *the study* in mathematics activity, *in the set of human activities and of social institutions*» (Chevallard, 1999).
This kind of analyses, although subjected to criticisms in D'Amore and Godino (2006, 2007), has opened the way to the use of anthropology as a critical instrument, as a new theoretical frame at research into mathematics education, in accordance with what has been already highlighted in the above quoted articles. It is the human being, strong of the acquired culture, strong of the specific

expressive, communicative luggage, who handles formal writings and gives them a meaning that it cannot be anything else but coherent with his social history; every meaning of each formal expression is the result of an anthropological comparison between a lived history and a here-and-now that must be coherent with that history.
We pass over this point, referring back to the quoted texts.

Psychology. In D'Amore and Godino (2006) we show how the shift from the anthropological picture to the onto-semiotic one is made necessary (amongst other things) by the need of not trivializing the presence of psychology in the study of learning and, in general, classroom situations. In D'Amore (1999) we show, for example, how ideas on representation drawn from psychology, regarding the explanation of the passage from image (weak) to model (stable) of concepts (Paivio, 1971; Kosslyn, 1980; Johnson-Laird, 1983; Vecchio, 1992), can be placed as a unitary basis of the explanation of several didactic phenomena, as intuitive models, the shift from internal to external models, the figural concepts, up to misconceptions, studied mainly in the 80s. Also the ideas of frame and script (Bateson, 1972; Schank, Abelson, 1977) have been used for the same purpose.

2. Changes of meaning: an analysis connecting theories within Mathematics Education.

A possible path we can follow to understand the phenomenon of "changes of meaning" is to network more than one semiotic approach (Santi, 2010). In this section, we present the issue of "changes of meaning" addressing two semiotic perspectives: Duval's structural and functional approach and Radford's cultural-semiotic approach. We show the complementarity of the two perspectives to give an encompassing interpretation of this didactical phenomenon. We use the connection of Duval's and Radford's perspectives to analyse a successful teaching experiment involving primary school pupils who do not change the sense of meaning when exposed to treatment transformations of figural representations of sequences.

2.1 A conceptual framework for changes of meaning

Duval's Structural and Functional Approach
Duval's (1995) approach stems from a realistic view point that considers mathematical objects a priori inaccessible ideal objects. Since mathematical objects are inaccessible entities, the theory pivots around the notion of semiotic systems and the coordination of semiotic systems through treatment and conversion transformations. A semiotic system is characterized by a set of elementary signs, a set of rules for the production and the transformation of signs and an underlying meaning structure deriving from the relationship between the signs within the system.
Mathematical objects, that cannot be referred to directly, are recognised as invariant entities that bind different semiotic representations as treatment and conversion transformations are performed. Duval identifies the specific cognitive functioning to mathematics with the coordination of a variety of semiotic systems. Both the development of mathematics as a field of knowledge and its learning are accomplished through such specific cognitive functioning.
Duval develops Frege's classical semiotic triangle (sinn-bedeutung-zeichen) and identifies meaning with the couple (sign-object), i.e. a relationship between a sign and the object it represents. The sign becomes a rich structure that condenses both the semiotic representation (zeichen) and the way the semiotic expression offers the object in relation to the underlying meaning of the semiotic structure sinn. Meaning therefore has a twofold dimension: sinn, the way a semiotic representation offers the object; bedeutung the reference to the inaccessible mathematical object. Meaning making processes

and learning require to handle different sinns networked through semiotic transformations without losing the bedeutung to the invariant mathematical object.
The following schema represents the construction of meaning when several semiotic systems are coordinated to conceptualize a mathematical object.

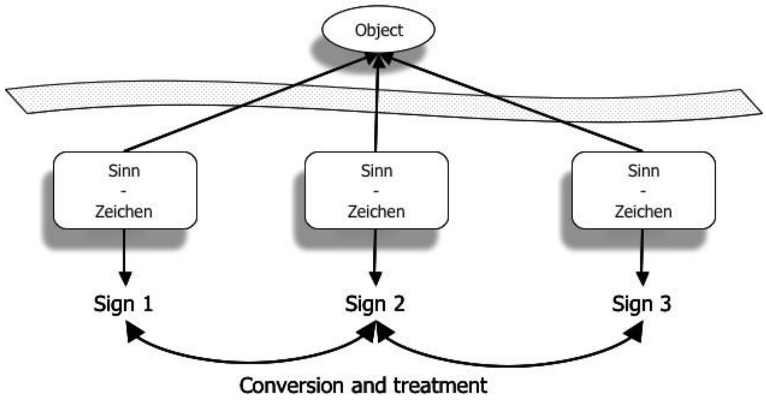


Fig. 1 Meaning and changes of meaning in Duval's approach

In this framework, the research issue is how students recognize the common bedeutung as the sinn changes through semiotic transformations. What we have above termed a "change of meaning", is a change of bedeutung as the sinn changes.

Radford's cultural-semiotic approach.
Within a socio-cultural and phenomenological standpoint, Radford's (2008) approach ascribes reflexive mediated activity, a central role both in cognition and in the emergence of mathematical objects. The reflexive activity entangles mathematical objects, semiotic resources, individuals' consciousness and intentional acts, within social practice and a cultural and historical dimension.
Mathematical objects are fixed patterns that emerge from the reflexive mediated activity. Mathematical objects lose any ideal and a priori existence but they are ontologically intertwined with the mediated activity from which they emerge. Nevertheless, mathematical objects acquire a form of ideality and existence in the culture that encompasses the reflexive activity.
Learning is considered an objectification process accomplished through a reflexive activity, a meaning making process that allows to become aware of the mathematical object that exists in the culture, but the student doesn't recognize. The complexity of the objectification process requires to broaden the notion of sign and go beyond its representational role, since signs culturally mediate activity and direct the individual's intention towards the mathematical object. Signs are termed as semiotic means of objectification and they include, artefacts, gestures, language, rhythm. Semiotic means of objectification stratify the mathematical object into levels of generality according the reflexive activity they mediate.

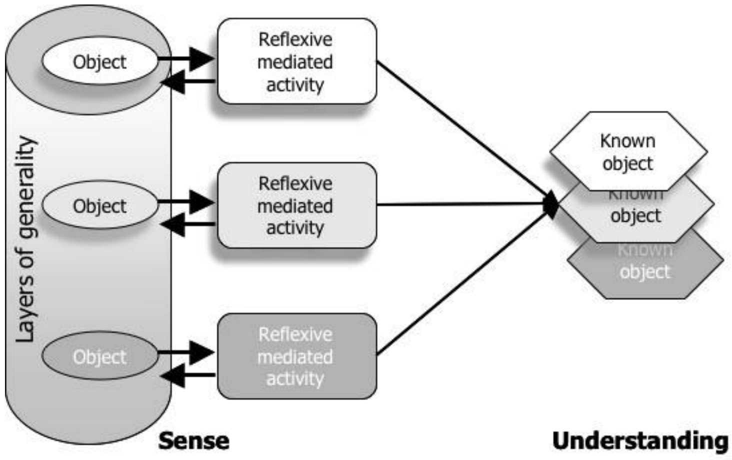


Fig. 2 Meaning and changes of meaning in Radford's perspectives

Meaning is no longer a mere relation sign-object, but is deeply interwoven with the reflexive activity, with intentional acts culturally mediated by semiotic means of objectification. Meaning is a double sided construct with a personal and a cultural dimension. The personal dimension refers to the individual's intentional acts directed towards a cultural unitary object. The cultural dimension refers to cultural and historical features that are condensed in the general and interpersonal mathematical object brought to the individual by teaching activities. The expected outcome of learning as an objectification process, is the alignment of the personal meaning with the cultural meaning.

Although both the theories we analysed are semiotic perspectives - if we look at the relationship with cognition - semiotics has a different hierarchical position in the respective system of principles. Therefore, the two theories have strong boundaries that separate them. This brings along also differences regarding the nature of mathematical objects and processes. In Duval's approach semiotics plays a representational role and it is the very substance of cognition that is identified with the coordination of semiotic systems. In Radford's perspective cognition is considered a process of objectification in which signs mediate a reflexive activity. Furthermore, the way signs are used is very different. In Duval's perspective, semiotic representations are used diachronically through treatment and conversion transformations. Whereas in Radford's perspective, a wide range of semiotic means of objectification are used synchronically organized around a particular mediator that changes as the level of generality changes. The different hierarchical position of semiotics allows Radford to broaden the notion of sign to include gestures, artefacts, rhythm, kinaesthetic activity etc. that Duval would never consider semiotic.

The different hierarchical position of semiotics stems from the different ontologies behind the two theories. The structural and functional approach has a realistic view of mathematical objects that ascribes to semiotics a representational role and to meaning a relation sign-object. The theory of objectification has a pragmatic standpoint towards mathematical objects that ascribes to semiotics the role of mediating a reflexive activity, the "substance" of ontology, meaning and cognition. Mathematical process are also differently positioned in the system of principle. Duval identifies the mathematical activity with the transformation of signs, subsumed in the robust structure of the semiotic systems that accomplish discursive and meta-discursive functions. Radford considers activity a form of reflection that involves the individual as a whole – his consciousness, feelings, perception, sensorimotor activity etc- immersed in a system of cultural signification that orients his intentional acts.

At a more profound level, any attempt to enlarge one of the theories subsuming elements of the other conflicts with its epistemological foundations. Nevertheless, the boundaries that separate the two theories do not imply an opposition between the two perspectives. Ullmann (1962) highlights

two complementary features that characterise the development of mathematical objects: the operational phase and the referential phase. On the one hand mathematical objects and their meaning emerge from and are objectified by a reflexive activity, on the other hand it is necessary to linguistically refer to the entities that emerge from such practices. The dual nature of mathematical objects – as patterns of activity and as "existing" ideal entities in the culture – implies that also meaning and semiotics have a dual nature. In the connecting theories terminology, the strong differences result in a high level of complementarity that accounts for networking by coordinating the two perspectives, respecting their identity. Coordinating Duval's and Radford's theories allows to encompass the double-sided nature of objects, meaning and representations.
The emergence of a mathematical object and its objectification is described by the cultural semiotic perspective whereas the reference to the object is accounted for by the structural and functional approach. Meaning as a sense making process of the individual and as the activity culturally condensed in the institutional object are described by Radford's approach; meaning as the interplay between sinn and an bedeutung is framed by Duval's approach. The coordination of the two theories is in turn achieved by the dual nature of semiotics. On the one hand signs mediate reflexive activity on the other hand they represent objects and broaden our cognitive possibilities through semiotic transformations. Our general conjecture is that a successful outcome of mathematical learning processes rests on the dual nature of semiotic resources, i.e. as semiotic registers and semiotic means of objectification; as a semiotic mean of objectification a sign -synchronically interwoven with a rich arsenal of mediators – supports the reflexive activity; a sign belonging to a semiotic system can be diachronically transformed into another to connote and denote mathematical objects. They are two complementary and interwoven aspects of the same phenomenon.

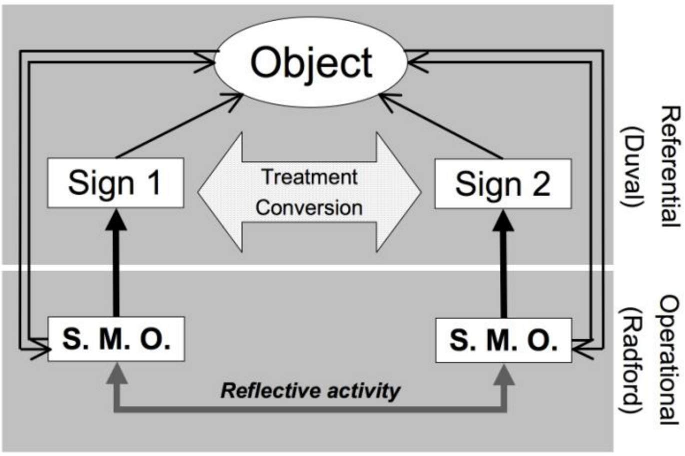


Fig. 3 The complementary roles of Duval's and Radford's approaches in framing the meaning of mathematical objects.

If we disregard signs as semiotic means of objectification, learning is an empty and meaningless manipulation of signs, if we disregard signs as belonging also to semiotic systems, mathematical objects wouldn't have developed into the form of rationality we know today and their conceptual acquisition would be impossible. The changes of meaning can be traced back to semiotic transformations that are not sustained by a mediated reflexive activity that guarantees the relation to the common cultural meaning of the mathematical object. The technology of the semiotic system allows the transformations of signs, but meaning in its cultural and personal sense evaporates, thereby losing also the correct interplay between sinn and bedeutung.

2.2 Analysis of a protocol

We present the protocol taken from an experimentation with primary school students working on sequences. According to a socio-cultural perspective students were immersed in a shared mathematical practice, working in small groups . We will analyse the sequence a_n=n^2+2n focussing on two different figural representations that are reported below.

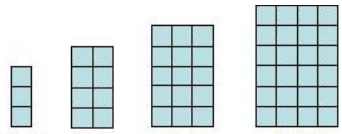


Fig. 1 Fig.2 Fig. 3 Fig. 4

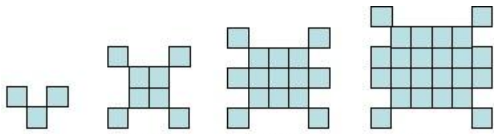
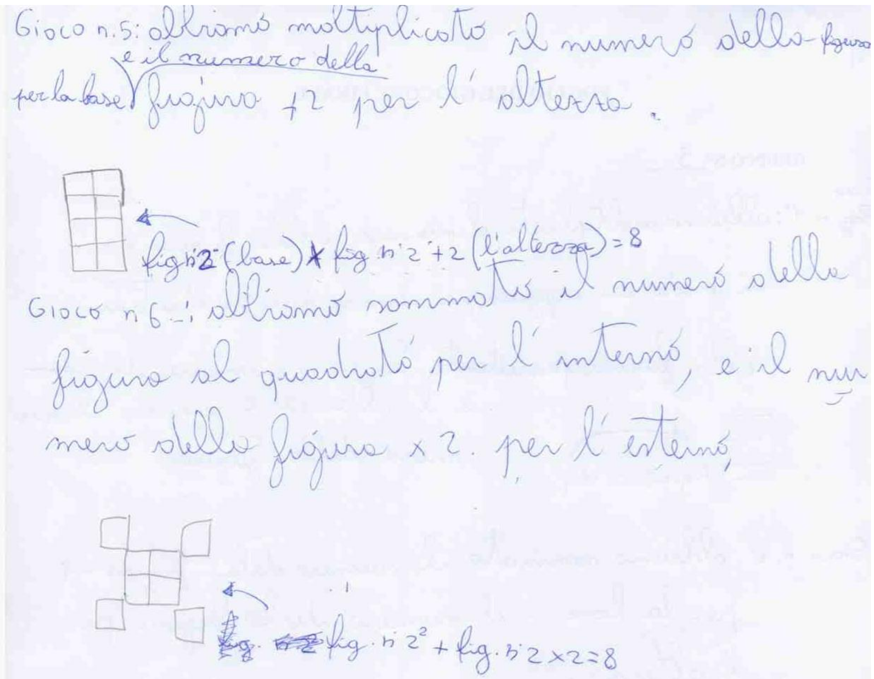


Fig. 1 Fig.2 Fig. 3 Fig. 4

Most of the groups that were able to determine the general schema, also recognized the same sequence as the figural representation changed. Without any explicit request, some students even attempted a first algebraic symbolism to express the general rule Video tapes testify also students' rich sensory-motor activity, conveyed mainly by gestures, that I cannot relate here but it is clearly condensed in the explanation of the two schemas where the generality of the rule is expressed with spatial-geometrical properties as base, height, inside, outside. Below the protocol of group 5 with the general schema to determine the number of elements of any figure of the sequence.



We multiplied the number of the figure for the base and the number of the figure +2 as the height.

We added the square of the number of the figure for inside and the number of the figure × 2 for outside.

The pupils were successful in finding the schema for the general term of the sequence and they didn't change meaning when exposed to treatment transformation in the figural register. To understand how students accomplish this result we have to use the conceptual framework we constructed above. Using semiotic resources both as semiotic means of objectification and semiotic registers, pupils grasp the dual nature of mathematical objects and their meaning.

Cultural-semiotic interpretation. Students objectify the mathematical sequence within the sociocultural space of the classroom. The use of semiotic means of objectification pivots around the figural representation that allows also the synchronic use of gestures and the sensorimotor activity. The activity was extremely meaningful to the students because it was intimately connected to their embodied experience. As the students are more and more involved in the reflexive activity there is an increasing agreement between the personal meaning and the cultural meaning of the mathematical object, thereby accessing higher levels of generality. This accounts for both the recognition of the same sequence, as the figural representation changes, and the spontaneous attempt to introduce a syncopated algebraic notation for the general term of the sequence.
Structural and functional interpretation. Students carry out a complicated network of semiotic transformations that involve both treatment and conversion. The task proposed to students, requires to connect three semiotic systems: the figural register, natural language and the arithmetical register. First of all, a very difficult conversion is necessary to construct the function that associates the number of elements in the figure to the number of the figure. Also to recognize the general schema of the sequence, students perform a conversion that involves the above registers; they first find the number of elements for a small number then they generalize the schema to a big number, thereby arriving to the general term of the sequence. The conversions are carried out passing the following order: figural register-arithmetical register (to calculate the number of elements in the figure)-natural language (to represent the general term). The outcome of the coordination of such semiotic systems is that students recognize the common reference (bedeutung) as the figural representations (sinn) changes.
Our contention is that students are able to handle meaning correctly at the referential level because the semiotic transformation is supported, at the operational level, by a strong embodied reflexive activity that involves the students consciousness within a sociocultural space of signification.

3. The teachers choices as a cause for misconceptions in the learning of the angle as a mathematical concept

Another research based on the *cultural-semiotic approach* is presented in Sbaragli and Santi (2011). Radford introduced the cultural-semiotic approach at the beginning of the 2000s and he ascribes to semiotics a central role within an anthropological viewpoint of mathematical objects and learning. This research shows how students' misconceptions on the concept of angle, highlighted by the broad literature on this subject, depend also on teachers' didactic choices relative to didactical transposition and didactical engineering. In particular, we focussed on the *incoherence of teachers' intentionality* with the cultural and conceptual aspects of the learning students should objectify. Such incoherences derive from a limited and unaware use of semiotics means of objectification.

3.1 The cultural-semiotic approach

Referring to the phenomenology of Edmund Husserl (1913/1959), Radford (2006) associates objectification, regarded as the attribution of meaning, to an *intentional act* which places the subject in relationship to the object of knowledge and provides a particular understanding of such object. When considering scientific knowledge, particularly in mathematics, we have to face the issue of the interpersonal and general nature of mathematical objects. The subjective and situated meaning of intentional acts does not fully encompass the generality that characterises scientific knowledge.
According to the cultural-semiotic approach that we are following, we cannot reduce our individual experience to a solitary sensory and cognitive interaction with the world, but the way in which we intentionally enter into contact with reality is intrinsically determined by historical and cultural factors. The mediators, the artefacts, the gestures, the symbols, and the words which Radford calls

semiotic means of objectification (Radford, 2003) are not only tools by which we manipulate the world, but mediators of our intentional acts, bearers of a historical consciousness built from the cognitive activity of the preceding generations. Such means determine and constitute the socially shared practices in which the processes of objectification develop.

Pupils and teachers find themselves immersed in a social and cultural context in which they find objects that are part of their culture. Institutionally, the teacher is in charge of guiding the pupil in the process of objectification, entrusting himself to the semiotic means of objectification and to the cultural ways of signification which culture and history have placed at his disposition.

It is useful for our analysis to take into account the fact that, according to Godino and Batanero (1994), and to D'Amore and Godino (2006), it is possible to attribute a personal and institutional dimension to the elements recalled above. The system of practices involves both a single individual and a group of institutionally recognised individuals, specifically the class. The same can be said for the mathematical object that exists both in a personal relationship with a subject and in an institutional relationship with the culture from which it emerged and with the social group that confers on it a knowledge value.

Learning, as a process of objectification requires an alignment between the personal dimension determined by the pupil's intentional acts and the institutional one that involves the historical and cultural aspects. The teaching-learning processes bring with them a dialectics between the personal aspects and the institutional ones bringing about the unification of the two dimensions towards a unified meaning. The construction of such a meaning, in which the unity of the individual with his culture is realised, is possible through the semiotic means of objectification that direct the intentional act of the individual towards the mathematical object. Such semiotic means, therefore, have their reason for being in that they are in the service of the intention of the individual and, at the same time, allow the embodying of knowledge and modes of rationality historically constructed by preceding generations. They, therefore, contribute to the creation of a shared meaning space that brings about the unity between the person and the culture, between personal meaning and institutional meaning, between individual intention and the object to which the intention is addressed.

It is necessary, therefore, to consider the complex network of individual and social practices, customs, beliefs, and convictions within which the teacher must daily orientate himself when he activates the mediators to encourage the learning of mathematical knowledge on the part of his pupils. This has to do with a network from which inconsistent behaviours can emerge on the part of the teacher.

It is from this point of view that it is possible to interpret the *avoidable misconceptions* (Sbaragli, 2005, pp. 56 and following) within the cultural semiotic perspective. In fact, such misconceptions depend directly on the choices of the teachers tied to the didactic transposition and the didactic engineering; two factors which, in the light of the cultural semiotic setting, turn out to be determining in the aligning of the personal meaning of the pupil and the cultural one, when the teacher manages the classroom practices.

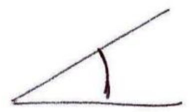
In particular Sbaragli and Santi (2011) focus their attention on the mathematical object "angle", highlighting the incoherence between the cultural meaning of the angle and the intentional acts objectified by the semiotic means chosen by some teachers to their pupils. The existence of incoherence can lead students to avoidable (misconceptions); misconceptions that from a semiotic point of view hinder a correct coordination of different representations, when they are giving sense to the mathematical object.

3.2. Researches on the angle

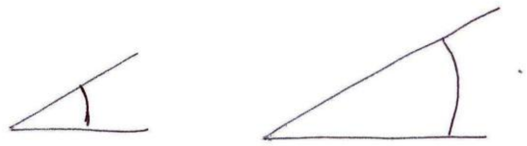
The research carried out by Sbaragli and Santi (2011) develops along two steps: the first is based on dialogues with 20 primary school teachers from different Italian teachers: the dialogues dealt with the concept of angle and their choices of the semiotic means of objectification they proposed to

their pupils. The dialogues developed from the researchers' questions that aimed at triggering a discussion to highlight teachers' convictions on the angle and their educational choices. The second step is based on questions regarding the conceptual questions posed to primary school pupils (grade K-5) of the teachers mentioned above. We interviewed 8 pupils taken from each class of the 20 teachers for a total of 160 students. The students were chosen at random draw. We interviewed these pupils to understand in depth their convictions on the angle.
This research singled out incoherence between the teachers' intentionality and the mathematical concept their students should objectify. Incoherence can be traced back to an unaware and limited use of semiotic means of objectification. We analyse an example of such an incoherent behaviour.

An example of incoherence. Of the 14 teachers out of 20 who stated that the angle is the part of a plane comprised between the two half-lines with the common origin, 9 choose as a semiotic means the arc near the origin of the angle which limits a part of the plane, 3 choose the part of the plane coloured up to the arc, and 2 direct their attention to the unlimitedness of the part of the plane.
The 12 teachers who choose to indicate the arc or to colour the part of the plane up to the arc placed importance, with such graphic semiotic means of objectification, on the limitedness of the part of the plane and not on its unlimitedness; unlimitedness is instead contemplated in their definition because the part of the plane deriving from such definition turns out to be 'open'.
After the interview, the choices of these 12 teachers were divided into two categories: 5 relative to the *lack of awareness of the mathematical knowledge they bring into play* and 7 relative to *the lack of a critical sense with respect to their own choice.*
We report a part of the interview regarding the incoherent due to *lack of awareness of the mathematical knowledge.*
R.: Why did you choose this representation?
C.: Because the angle is represented like this.
R.: In what sense is it represented like this?
C.: When you want to talk about an angle, you draw it like this:



and the children know that we are talking about an angle.
In terms of the cultural-semiotic approach the there is no synchronic use of semiotic means of objectification. The restriction to the "little arc" fixes at a strong embodied level an incorrect objectification of the mathematical object keeping the student away from a rich mathematical activity that traces back the historical and cultural evolution of the mathematical object.
Note how this choice appears univocal in the eyes of that teacher.
The interview continues in the following way:
R.: Indicate, on this illustration, which angle you are speaking about.
(C. He indicates the part of the plane up to the arc).
R.: Up to where does the angle arrive?
C.: Up to here (he indicates the arc).
R.: Can you go beyond this arc?
C.: No, it goes up to here.
R.: Can't we go beyond the arc?
C.: In this case, no.
R.: And in which cases can we go beyond?
C.: If the angle is bigger.
(He draws another angle, apparently of the same amplitude, with longer half-lines and arc).



From this extract it emerges how misconceptions about the angle deriving from graphic representations, confirmed by classical research in the field and described in literature are present in some cases in the teachers themselves and therefore transferred to their pupils. The use of the “little arc” hinders the unlimited meaning of the angle that can be grasped at a higher level of generality that goes beyond the embodied meaning conveyed by this figural representation. The synchronic use of other semiotic means of objectification would allow to overcome this limit and access a disembodied meaning of this mathematical object.
The interview continued in the following way:
R.: Why did you choose this representation?
C.: Because this is the way to represent the angle.
R.: It is the way chosen by whom?
C.: By everyone, in all the books, it is like this.
R.: And do you like this representation?
C.: Yes, I have always done it this way, I don’t see why I should change it.
R.: What, for you, is an angle?
C.: It is the part of the plane comprised between two half-lines that start from the same point.
R.: And how is this part of the plane?
C.: In what sense?
R.: What properties does this part of the plane have?
C.: I don’t understand.
R.: Is this part of the plane of which you are speaking limited or unlimited?
C. He looks at his drawing, thinks a bit and then answers:
C.: It is limited by the half-lines.
R.: And here, how is it? (The researcher indicates the unlimited part of the plane).
C.: It arrives up to here (he indicates the arc).
R.: When I asked you what an angle is, why didn’t you say that it arrives up to the arc?
C.: Because it isn’t mentioned in the definition, but it becomes evident in the drawing.
We highlight that the graphic semiotic means of objectification is in contrast with the cultural meaning of the object conveyed by the verbal definition that the teacher expects her students to learn.
We highlight also that that the answer of the 160 selected students described in Sbaragli and Santi (2011) are not connected with the cultural and conceptual learning objectives of their teachers; in particular the graphic semiotic mean proposed by the teacher is stronger than her cultural and conceptual objective. In some cases, the graphic semiotic prevails to such an extent that it distorts the teacher’s intention; for example when the extension of the angle is identified with the length of the little arc the little arc itself. Students confuse the graphic representation with the concept proposed by the teacher. Furthermore there are students’ answers unexpected by their teachers deriving by everyday natural language (angle as synonymous of vertex) and a limited interpretation of the limited interpretation of the few and sometimes unique semiotic means of objectification proposed in the classroom.
The individual’s (the teacher) intentionality plays a crucial role in the possibility of ascribing meaning to the mathematical object. Such an intentionality should be handled with awareness to be educationally effective. Referring to Husserl (1913/1959), the results of this research highlight that the teacher, in classroom practices, too often creates inconsistency between the intentional act that determines the way in which the object is presented to consciousness (noesis) and the conceptual

content of the individual experience (noema). Consistency and unity of the different intentional acts of the teacher do not seem to be always present in the classroom practices, when dealing with the angle.

In fact, the inconsistency between the explicit intentionality of the teacher, through verbal means of objectification, and the graphicones, chosen to express this concept, can be the source of avoidable misconceptions in the mind of the pupil. The choice of the signs is not, in fact, neutral or independent. Radford (2005b, page 204) claims that «semiotic means of objectification offer several possibilities for carrying out a task, designating objects and expressing intentions. (…) It is necessary, therefore, to know how to identify the semiotic means of objectification to obtain objects of consciousness», such an identification should be managed with a strong critical sense on the part of the teacher.

Semiotic means of objectification should not be considered as *a priori* choices, that stem from outside the classroom without a critical analysis on the part of the teacher. To overcome *avoidable misconceptions* it is therefore essential to provide a variety of semiotic means that allow objectification processes within a social system of signification handled by teachers with awareness.

References

Bateson G. (1972). *Steps to an Ecology of Mind*. Novato Ca.: Chandler Publishing Company.

Chevallard Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*. 19, 2, 221-266.

D'Amore B. (2003). *Le basi filosofiche, pedagogiche, epistemologiche e concettuali della didattica della matematica*. Bologna: Pitagora. Portuguese edition (Sao Pãolo: Escrituras, 2005) and Spanish edition (Barcelona-México: Reverté, 2005).

D'Amore B. (2005). Pratiche e metapratiche nell'attività matematica della classe intesa come società. Alcuni elementi rilevanti della didattica della matematica interpretati in chiave sociologica. *La matematica e la sua didattica.* 3, 325-336.

D'Amore B. (2006). Objetos, significados, representaciones semióticas y sentido. In: L. Radford and B. D'Amore (eds.) (2006). *Semiotics, Culture and Mathematical Thinking*. Special issue of the journal *Relime*. 177-196.

D'Amore B. (2007). How the treatment or conversion changes the sense of mathematical objects [Invited speaker article]. In: E.P. Avgerinos and A. Gagatsis (eds.) (2007). *Current trends in Mathematics Education*. Proceedings of 5th MEDCONF 2007 (Mediterranean Conference on Mathematics Education), 13-15 April 2007, Rhodes, Greece. Athens: New Technologies Publications. 77-82. ISBN: 978-960-89713-0-1.

D'Amore B., Fandiño Pinilla MI. (2001). Concepts et objects mathématiques. In: Gagatsis A. (ed.) (2001). *Learning in Mathematics and Science and Educational Technology*. Atti del Third Intensive Programme Socrates-Erasmus, Nicosia, Università di Cipro, 22 Giugno – 6 luglio 2001. Nicosia (Cjprus): Intercollege. 111-130.

D'Amore B., Fandiño Pinilla M.I. (2007a). How the sense of mathematical objects changes when their semiotic representations undergo treatment and conversion. *La matematica e la sua didattica*. Vol. 21, n. 1, 87-92. Proceedings of : Joint Meeting of UMI-SIMAI/SMAI-SMF: *Mathematics and its Applications*. Panel on Didactics of Mathematics. Department of Mathematics, University of Turin. July 6, 2006.

D'Amore B., Fandiño Pinilla M.I. (2007b). Change of the meaning of mathematical objects due to the passage between their different representations. How other disciplines can be useful to the analysis of this phenomenon. Rome, *Symposium on the occasion of the 100th anniversary of ICMI*, March 2008. WG5: *The evolution of theoretical framework in mathematics education*, organizers: Gilah Leder and Luis Radford. www.unige.ch/math/EnsMath/Rome2008

D'Amore B., Godino D.J. (2006). Punti di vista antropologico ed ontosemiotico in Didattica della Matematica. *La matematica e la sua didattica*. 1, 9-38.
D'Amore B., Godino D.J. (2007). El enfoque ontosemiótico como un desarrollo de la teoría antropológico en didáctica de la matemática. *Relime*. 10, 2, 191-218.
Duval R. (1995). *Sémiosis et pensée humaine. Registres sémiotiques et apprentissages intellectuels*. Berne: Peter Lang.
Ernest P. (1991). *The philosophy of mathematics education*. London: Falmer Press.
Husserl E. (1931). *Ideas*. London (U. K.): The Macmillan Company.
Husserl E. (1961). *Recherches logiques*. (Recherches I et II). Paris: PUF.
Husserl E. (1913/1959). *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*. Dordrecht: Kluwer.
Godino J., Batanero C. (1994). Significado institucional y personal de los objectos matemáticos. *Recherches en Didactique des Mathématiques*. 14, 3, 325-355.
Ilyenkov E.V. (1977). *Dialectical Logic*. Moscow: Progress Publishers.
Johnson-Laird P.N. (1983). *Mental Models*. Cambridge Ma.: Cambridge Univ. Press.
Kosslyn S.M. (1980). *Image and Mind*. Cambridge Ma.: Harvard Univ. Press.
Kutschera F. von (1979). *Filosofia del lenguaje*. Madrid: Gredos.
Paivio A. (1971). *Imagery and Verbal Processes*. New York: Holt, Rinehart and Winston.
Radford L. (2002). On heroes and the collapse of narratives: a contribution on the study of symbolic thinking. In: A.D. Cockburn and E. Nardi (eds.) (2002). *Proceedings of the 16th Conference of the International Group for the Psychology of Mathematics Education, PME 26*. 4, 81-88.
Radford L. (2003). Gestures, speech and the sprouting of signs. *Mathematics Thinking and Learning*. 5(1), 37-70.
Radford L. (2004a). Cose sensibili, essenze, oggetti matematici ed altre ambiguità. *La matematica e la sua didattica*. 1, 4-23.
Radford L. (2004b). La généralisation mathématique comme processus sémiotique. In: G. Arrigo (ed.) (2004). *Proceedings of the Ticino Mathematics Education Conference.* Quaderni Alta Scuola Pedagogica. Bellinzona (Switzerland): Centro didattico cantonale. 11-27.
Radford L. (2005). La generalizzazione matematica come processo semiotico. *La matematica e la sua didattica.* 2, 191-213.
Radford L. (2006). The Anthropology of Meaning. *Educational Studies in Mathematics*. 61, 39-65.
Radford L (2008). A cultural theory of learning. In L.Radford, G. Schubring, & F. Seeger (eds.) *Semiotics in Mathematics Education*. Rotterdam: Sense Publishers. 215-234.
Santi G. (2010). *Changes in meaning of mathematical objects due to semiotic transformations: a comparison between semiotic perspectives*. PhD dissertation. University of Palermo.
Russell B. (1976). *An Inquiry into Meaning and Truth*. London: G. Allen and Unwin.
Sbaragli S. (2005). Misconcezioni "inevitabili" e misconcezioni "evitabili". *La matematica e la sua didattica.* 1, 57-71.
Sbaragli S., Santi G. (2011). Teacher's choices as the cause of misconceptions in the learning of the concept of angle. On referee.
Schank R.C., Abelson R.P. (1977). *Scripts plans goals and understanding*. Hillsdale N.J.: Lea.
Speranza F. (1997). *Scritti di Epistemologia della Matematica*. Bologna: Pitagora.
Ullmann, S. (1962). *Semántica. Introducción a la ciencia del significado*. Madrid: Aguilar
Vecchio L. (1992). Breve storia delle ricerche sull'immagine mentale. In: L. Vecchio (ed.) (1992). *Le immagini mentali*. Proceedings of the homonymous Meeting of Pavia (October, 1990). Firenze, Italy: La Nuova Italia. 15-48.