Decomposable quadratic forms and involutions

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Abstract

In his book on compositions of quadratic forms, Shapiro asks whether a quadratic form decomposes as a tensor product of quadratic forms when its adjoint involution decomposes as a tensor product of involutions on central simple algebras. We give a positive answer for quadratic forms defined over local or global fields (or, more generally, over any linked field) and produce counterexamples over fields of rational fractions in two variables over any formally real field.

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1. Introduction

Every nondegenerate symmetric or skew-symmetric bilinear form \( b \) on a finite-dimensional vector space \( V \) induces an involution (i.e. an anti-automorphism of period 2)
on the endomorphism algebra $\text{End} V$. This involution, known as the adjoint involution of $b$ and denoted by $\text{ad}_b$, is characterized by the following property:

$$b(x, f(y)) = b(\text{ad}_b(f)(x), y) \quad \text{for } x, y \in V \text{ and } f \in \text{End} V.$$  

If the form $b$ is the tensor product of two nondegenerate symmetric or skew-symmetric bilinear forms, i.e.

$$V = V_1 \otimes V_2 \quad \text{and} \quad b = b_1 \otimes b_2,$$

then it is easy to see that the adjoint involution also decomposes,

$$\text{End} V = (\text{End} V_1) \otimes (\text{End} V_2) \quad \text{and} \quad \text{ad}_b = \text{ad}_{b_1} \otimes \text{ad}_{b_2},$$

(see [11, Corollary 6.10]), which we denote

$$(\text{End} V, \text{ad}_b) = (\text{End} V_1 \otimes \text{End} V_2, \text{ad}_{b_1} \otimes \text{ad}_{b_2}).$$

In [11, p. 201], Shapiro raises the following question: if the adjoint involution has a non-trivial decomposition of the form

$$(\text{End} V, \text{ad}_b) \simeq (A_1, \sigma_1) \otimes (A_2, \sigma_2)$$

for certain involutions $\sigma_1, \sigma_2$ on central simple algebras $A_1, A_2$, does it follow that $b$ decomposes nontrivially into a tensor product of bilinear forms? He showed that the answer is positive if $A_1$ and $A_2$ are split, see [11, Corollary 6.10] or Proposition 3.1 below.

In this paper, we give an affirmative answer to Shapiro’s question over certain fields of characteristic different from 2, including local and global fields, and give examples where the answer is negative. Our examples are involution trace forms of central simple algebras of degree $2m$, where $m$ is an odd integer, see Section 3.2.

Throughout the paper, the characteristic of the base field $F$ is assumed to be different from 2. Therefore, symmetric bilinear forms correspond bijectively to quadratic forms, and this correspondence is used to define the tensor product of quadratic forms. We simply denote by $q_1 q_2$ the tensor product of the quadratic forms $q_1$ and $q_2$. For any (nondegenerate) quadratic form $q$, we write $\text{ad}_q$ for the adjoint involution of the polar form of $q$.

In Section 2, we collect results on factorizations of quadratic forms. In particular, we determine the decomposable forms of dimension at most 8. The decomposition of involutions is discussed in Section 3.

2. Decomposable quadratic forms

All the quadratic forms considered in this paper are nondegenerate. A quadratic form $q$ over a field $F$ is called decomposable if there is a factorization $q \simeq q_1 q_2$ where $q_1$
and $q_2$ are quadratic forms with $\dim q_1, \dim q_2 \geq 2$. In particular, the hyperbolic forms of dimension $n \geq 4$ are decomposable since they are isometric to

$$\langle 1, -1 \rangle \otimes \left( \frac{\theta}{2} \langle 1 \rangle \right).$$

From the outset, note that the notion of decomposable form is not compatible with Witt equivalence, since every quadratic form $q$ is Witt-equivalent to the decomposable form $q \otimes \langle 1, 1, -1 \rangle$.

After reviewing below some general properties of factorizations of quadratic forms, we consider factorizations of forms of small dimension, and determine the indecomposable forms over some special fields.

2.1. General properties

**Lemma 2.1.** If $q_1$, $q_2$ are even-dimensional quadratic forms, the discriminant and Clifford invariant of the product $q_1q_2$ satisfy

$$\text{disc}(q_1q_2) = 1 \quad \text{and} \quad c(q_1q_2) = (\text{disc } q_1, \text{disc } q_2)_F.$$

**Proof.** For $i = 1, 2$, let $d_i \in F^\times$ be such that $\text{disc } q_i = d_i F^{x^2}$. Since $q_i$ is even-dimensional, we have $q_i \equiv \langle 1, -d_i \rangle \mod I^2 F$, hence

$$q_1q_2 \equiv \langle 1, -d_1 \rangle \langle 1, -d_2 \rangle \mod I^3 F.$$

Therefore,

$$\text{disc}(q_1q_2) = \text{disc}(\langle 1, -d_1 \rangle \langle 1, -d_2 \rangle) = 1$$

and

$$c(q_1q_2) = c(\langle 1, -d_1 \rangle \langle 1, -d_2 \rangle) = (d_1, d_2)_F. \quad \Box$$

For any quadratic form $q$ over $F$, let $q_a$ denote the anisotropic kernel of $q$, and let

$$\dim_a q = \dim q_a.$$

**Lemma 2.2.** If a quadratic form $q$ satisfies $\dim q = n_1n_2$ for some integers $n_1, n_2$ subject to

$$n_1 = \dim_a q \mod 2, \quad n_2 = 1 \mod 2,$$

$$n_1 \geq \dim_a q, \quad n_1, n_2 \geq 2,$$

then $q$ is decomposable.
Proof. Letting \( k_1 = \frac{1}{2}(n_1 - \dim_a q) \) and \( k_2 = \frac{1}{2}(n_2 - 1) \), we have
\[
q = (q_a \perp k_1(1, -1)) \otimes (1 \perp k_2(1, -1)).
\]

Besides these (essentially trivial) factorizations, a main source of decompositions is the following well-known result (due to Pfister):

**Lemma 2.3.** An anisotropic quadratic form \( q \) splits over a quadratic extension \( F(\sqrt{d}) \) (i.e., becomes hyperbolic over \( F(\sqrt{d}) \)) if and only if \( q \simeq \langle 1, -d \rangle \otimes q' \) for some quadratic form \( q' \).

**Proof.** See [10, Chapter 2, Theorem 5.2]. \( \square \)

**Corollary 2.4.** Let \( q \) be a decomposable quadratic form of dimension \( \dim q = 2p \) for some odd prime number \( p \). If \( \text{disc} q = 1 \), then \( q \) is hyperbolic. If \( \text{disc} q \neq 1 \), then \( \dim_a q \equiv 2 \mod 4 \) and \( q \) splits over \( F(\sqrt{\text{disc} q}) \).

**Proof.** Let \( q = q_1q_2 \) be a nontrivial decomposition of \( q \). We may assume \( \dim q_1 = 2 \) and \( \dim q_2 = p \). Comparing discriminants, we obtain \( \text{disc} q = \text{disc} q_1 \) since \( p \) is odd. Therefore, if \( \text{disc} q = 1 \), then \( q_1 \) is hyperbolic, hence \( q \) also is hyperbolic. If \( \text{disc} q = dF^{\times 2} \neq 1 \), then \( F(\sqrt{d}) \) splits \( q_1 \), hence also \( q \) and \( q_a \). By Lemma 2.3, it follows that \( q_a \simeq \langle 1, -d \rangle \otimes q' \) for some quadratic form \( q' \). If \( \dim q' \) is even, then \( \text{disc} q_a = 1 \), a contradiction since \( \text{disc} q = \text{disc} q_a \). Therefore, \( \dim_a q \equiv 2 \mod 4 \). \( \square \)

The results above yield necessary and sufficient conditions for the decomposability of forms \( q \) with \( \dim a q \leq 4 \), as we now show.

**Proposition 2.5.** A quadratic form \( q \) of odd dimension is decomposable whenever its dimension has a nontrivial factorization \( \dim q = n_1n_2 \) with \( n_1 \geq \dim_a q \). In particular, if \( \dim_a q \leq 3 \), then \( q \) is decomposable if and only if \( \dim q \) is neither 1 nor a prime number.

**Proof.** If \( \dim q \) is odd, then \( \dim a q \) is odd and the conditions (1) hold for any factorization \( \dim q = n_1n_2 \). The proposition follows from Lemma 2.2. \( \square \)

**Proposition 2.6.** If \( \dim a q = 2 \), then \( q \) is indecomposable if and only if \( \dim q \) is a power of 2.

**Proof.** Note that \( \dim q \equiv \dim a q \mod 2 \), so the hypothesis implies \( \dim q \) is even. If it has an odd prime factor \( p \), let \( n_1 = (\dim q)/p \) and \( n_2 = p \). Then \( n_1 \) is even, so \( q \) is decomposable by Lemma 2.2. Therefore, \( q \) is decomposable if \( \dim q \) is not a power of 2.

Conversely, if \( \dim q \) is a power of 2, then every nontrivial factorization of \( q \) has the form \( q = q_1q_2 \) with \( \dim q_1, \dim q_2 \) even. Then \( \text{disc} q = 1 \) by Lemma 2.1, which is impossible since \( \dim a q = 2 \). \( \square \)

**Proposition 2.7.** If \( \dim a q = 4 \), then \( q \) is indecomposable if and only if one of the following conditions holds:
(a) \( \dim q = 2p \) for some odd prime \( p \),
(b) \( \dim q \) is a power of 2 and \( \text{disc } q \neq 1 \).

**Proof.** Since \( \dim_a q = 4 \), it follows that \( \dim q \) is even. In case (a), \( q \) is indecomposable by Corollary 2.4. If \( \dim q \) is a power of 2, then every nontrivial factorization of \( q \) is into a product of forms of even dimension. The existence of such a factorization implies \( \text{disc } q = 1 \) by Lemma 2.1, hence \( q \) is indecomposable in case (b). Conversely, Lemma 2.2 shows that \( q \) is decomposable whenever \( \dim q \) has a factorization \( \dim q = n_1n_2 \) with \( n_1 \) even, \( n_2 \) odd and \( n_1 \geq 4, n_2 \geq 3 \). Since \( \dim q \) is even, such a factorization exists unless \( \dim q \) is a power of 2 or of the form \( 2p \) for some odd prime \( p \). To prove that either (a) or (b) holds when \( q \) is indecomposable, it only remains to show that \( q \) is decomposable if \( \dim q \) is a power of 2 and \( \text{disc } q = 1 \). In this case, \( q_a \) has a diagonalization

\[
q_a = \langle a, b, c, d \rangle
\]

with \( abcd \in F^\times \), so

\[
q_a = \langle a, b \rangle \otimes \langle 1, ac \rangle
\]

and, for \( k = \frac{1}{4}(\dim q) - 1 \),

\[
q = \langle a, b \rangle \otimes (\langle 1, ac \rangle \perp k(1, -1)) \tag{\*}
\]

2.2. **Forms of small dimension**

We may also use Propositions 2.6 and 2.7, together with Corollary 2.4, to discuss the decomposability of forms of dimension at most 8. Of course, quadratic forms whose dimension is 1 or a prime number are indecomposable. Therefore, we need to consider only forms of dimension 4, 6 or 8.

**Proposition 2.8.** A quadratic form \( q \) of dimension 4 is indecomposable if and only if one of the following conditions holds:

(a) \( \dim_a q = 2 \),
(b) \( \dim_a q = 4 \) and \( \text{disc } q \neq 1 \).

**Proof.** This readily follows from Propositions 2.6 and 2.7. \( \Box \)

**Proposition 2.9.** A quadratic form \( q \) of dimension 6 is indecomposable if and only if one of the following conditions holds:

(a) \( \dim_a q = 4 \),
(b) \( \dim_a q = 6 \) and \( \text{disc } q = 1 \),
(c) \( \dim_a q = 6, \text{disc } q \neq 1 \) and \( q \) is not split by \( F(\sqrt{\text{disc } q}) \).
Proof. The form \( q \) is decomposable if \( \dim_a q = 2 \), by Proposition 2.6, and indecomposable if \( \dim_a q = 4 \), by Proposition 2.7. Therefore, it only remains to consider the case where \( \dim_a q = 6 \), i.e., \( q \) is anisotropic. Corollary 2.4 shows that \( q \) is indecomposable in cases (b) and (c), and Lemma 2.3 proves that \( q \) is decomposable if \( \text{disc} q \neq 1 \) and \( q \) is split by \( F(\sqrt{\text{disc} q}) \). □

Proposition 2.10. A quadratic form \( q \) of dimension 8 is indecomposable if and only if one of the following conditions holds:

(a) \( \dim_a q = 2 \),
(b) \( \dim_a q = 4 \) and \( \text{disc} q \neq 1 \),
(c) \( \dim_a q = 6 \),
(d) \( \dim_a q = 8 \) and \( q \) is not split by any quadratic extension of \( F \).

Proof. If \( \dim_a q = 2 \), then \( q \) is indecomposable by Proposition 2.6. If \( \dim_a q = 4 \), then \( q \) is indecomposable if and only if \( \text{disc} q \neq 1 \), by Proposition 2.7. Suppose next \( \dim_a q = 6 \). If \( \text{disc} q \neq 1 \), then \( q \) is not decomposable by Lemma 2.1. If \( \text{disc} q = 1 \), then \( q \) is Witt-equivalent to an anisotropic Albert form, hence its Clifford invariant has index 4, by [5]. On the other hand, forms which decompose into a product of forms of even dimension have a Clifford invariant of index at most 2 by Lemma 2.1; therefore \( q \) is indecomposable. Finally, the case where \( \dim_a q = 8 \) follows from Lemma 2.3. □

2.3. Forms over special fields

Recall from [10, Chapter 2, §16] that the \( u \)-invariant of a field \( F \) is the supremum of the dimensions of anisotropic quadratic forms over \( F \).

Corollary 2.11. If \( u(F) \leq 4 \), a quadratic form \( q \) over \( F \) is indecomposable if and only if one of the following conditions holds:

(a) \( \dim q \) is a prime number,
(b) \( \dim_a q = 2 \) and \( \dim q \) is a power of 2,
(c) \( \dim_a q = 4 \) and \( \dim q = 2p \) for some odd prime \( p \),
(d) \( \dim_a q = 4 \), \( \dim q \) is a power of 2 and \( \text{disc} q \neq 1 \).

In particular, every quadratic form \( q \) with \( \dim q \equiv 0 \mod 4 \) and \( \text{disc} q = 1 \) is decomposable.

Proof. The hypothesis \( u(F) \leq 4 \) implies \( \dim_a q \leq 4 \). The corollary therefore readily follows from Propositions 2.5, 2.6 and 2.7. For the last statement, observe that \( \dim_a q \neq 2 \) if \( \text{disc} q = 1 \). □

The corollary applies for instance to \( p \)-adic fields or to nonformally real number fields. Note that case (d) does not arise for \( p \)-adic fields since every anisotropic quadratic form of dimension 4 then has trivial discriminant.
Suppose now $F$ is a real-closed field. Quadratic forms over $F$ are classified by their dimension and signature, which are related by

$$|\operatorname{sgn} q| \leq \dim q \quad \text{and} \quad \operatorname{sgn} q \equiv \dim q \mod 2.$$  

**Proposition 2.12.** A quadratic form $q$ over a real-closed field $F$ is decomposable if and only if $\dim q$ and $\operatorname{sgn} q$ have factorizations of the form

$$\dim q = n_1 n_2, \quad \operatorname{sgn} q = s_1 s_2$$

with

$$n_i \geq 2, \quad |s_i| \leq n_i \quad \text{and} \quad s_i \equiv n_i \mod 2 \quad \text{for} \ i = 1, 2.$$

**Proof.** If $q = q_1 q_2$ is a nontrivial decomposition, then

$$\dim q = \dim q_1 \dim q_2 \quad \text{and} \quad \operatorname{sgn} q = \operatorname{sgn} q_1 \operatorname{sgn} q_2$$

are factorizations of the required form. Conversely, given factorizations as in the statement of the proposition, one may find quadratic forms $q_1, q_2$ over $F$ with $\dim q_i = n_i$ and $\operatorname{sgn} q_i = s_i$. The forms $q$ and $q_1 q_2$ have the same dimension and signature, hence $q = q_1 q_2$.  

**Corollary 2.13.** If a quadratic form $q$ over a real-closed field $F$ satisfies

$$\dim q \equiv 0 \mod 4 \quad \text{and} \quad \operatorname{disc} q = 1,$$

then $q = (1, 1) \otimes q'$ for some quadratic form $q'$. In particular, $q$ is decomposable.

**Proof.** Since $\dim q$ is even and $\operatorname{disc} q = 1$, it follows that $\operatorname{sgn} q \equiv 0 \mod 4$. Therefore, we may apply the proposition above with $n_1 = s_1 = 2$.  

The case of quadratic forms of dimension divisible by 4 and trivial discriminant is particularly relevant for Shapiro’s question, see Section 3. The following result (of which Corollary 2.13 is a special case) gives further examples where these forms have a nontrivial decomposition.

**Proposition 2.14.** Suppose the field $F$ is linked, i.e., every tensor product of quaternion $F$-algebras is Brauer-equivalent to a quaternion algebra. Then every quadratic form $q$ over $F$ such that $\dim q \equiv 0 \mod 4$ and $\operatorname{disc} q = 1$ decomposes as $q = q_1 q_2$ for some quadratic forms $q_1, q_2$ with $\dim q_1 = 2$.

The proposition applies in particular to local and global fields, fields of transcendence degree at most 1 over a real-closed field, and fields of transcendence degree at most 2 over an algebraically closed field. We are indebted to Detlev Hoffmann, who pointed out that
the main theorem in [3] could extend our results and considerably simplify our original approach.

**Proof.** The proposition is clear if $q$ is hyperbolic. Suppose $q$ is not hyperbolic, and consider its Witt decomposition $q = q_\perp m (1, -1)$. The hypotheses on $q$ show that $q_\perp$ lies in $I^2 F$, hence by [3] there is a factorization $q_\perp = q_1 q'$ with $\dim q_1 = 2$. If $m$ is odd, then

$$\text{disc } q_\perp = \text{disc } q_1,$$

hence the condition $\text{disc } q = 1$ implies $q_1$ is hyperbolic, a contradiction. Therefore, $m$ is even and we may write $q = q_1 q_2$ with

$$q_2 = q' \perp \frac{m}{2} (1, -1).$$

**Remark.** For an arbitrary field $F$, the arguments above also show that every anisotropic form in $I^2 F$ has a 2-dimensional factor if every quadratic form $q$ with $\dim q \equiv 0 \mod 4$ and $\text{disc } q = 1$ decomposes as $q = q_1 q_2$ for some forms $q_1, q_2$ with $\dim q_1 = 2$. The theorem of [3] then shows that $F$ is linked, proving a converse of Proposition 2.14.

3. Decomposable adjoint involutions

Let $q$ be a quadratic form on an $F$-vector space $V$ and let $\text{ad}_q$ be the adjoint involution on $\text{End}_F V$. In this section, we consider decompositions of algebras with involution

$$(\text{End}_F V, \text{ad}_q) \simeq (A_1, \sigma_1) \otimes_F (A_2, \sigma_2)$$

(3)

where $A_1, A_2$ are central simple $F$-algebras and $\sigma_1, \sigma_2$ are involutions on $A_1$ and $A_2$, respectively. The elements in $F$ are fixed under $\text{ad}_q$, hence also under $\sigma_1$ and $\sigma_2$. Therefore, we only consider involutions which restrict to the identity on the center (i.e., involutions of the first kind). Since $\text{End}_F V$ represents the trivial element in the Brauer group of $F$, the Brauer classes of $A_1$ and $A_2$ are opposite. However, $\sigma_1$ yields an isomorphism between $A_1$ and its opposite algebra, hence $A_1$ and $A_2$ are also Brauer-equivalent. Since $\text{ad}_q$ is an orthogonal involution, the involutions $\sigma_1$ and $\sigma_2$ must be both orthogonal or both symplectic, by [11, Proposition 6.9] or [5, (2.23)].

In [11, p. 201], Shapiro asks whether a nontrivial decomposition as in (3) implies that $q$ is decomposable.

3.1. Positive results on Shapiro’s question

The case where $A_1$ and $A_2$ are split and $\sigma_1, \sigma_2$ are orthogonal is discussed in Shapiro’s book [11]:

**Proposition 3.1.** Let $q, q_1, q_2$ be quadratic forms on $F$-vector spaces $V, V_1, V_2$. The following statements are equivalent:
(a) \((\text{End}_F V, \text{ad}_q) \cong (\text{End}_F V_1, \text{ad}_{q_1}) \otimes_F (\text{End}_F V_2, \text{ad}_{q_2})\);

(b) there exists \(\lambda \in F^\times\) such that \(q \cong (\lambda)q_1q_2\).

**Proof.** See [11, Corollary 6.10, p. 112].

**Corollary 3.2.** Suppose that one of the algebras \(A_1, A_2\) in (3) is split. Then \(A_1\) and \(A_2\) are both split and the quadratic form \(q\) is decomposable if \(\text{deg } A_1, \text{deg } A_2 > 1\). Moreover, if \(\sigma_1\) and \(\sigma_2\) are symplectic, then \(q\) is hyperbolic.

**Proof.** The first part is clear, since \(A_1\) and \(A_2\) are Brauer-equivalent. Let \(A_1 = \text{End}_F V_1\) and \(A_2 = \text{End}_F V_2\) for some \(F\)-vector spaces \(V_1, V_2\). If \(\sigma_1\) and \(\sigma_2\) are orthogonal, then there exist quadratic forms \(q_1\) on \(V_1\) and \(q_2\) on \(V_2\) such that \(\sigma_1 = \text{ad}_{q_1}\) and \(\sigma_2 = \text{ad}_{q_2}\).

By Proposition 3.1, it follows from (3) that \(q = (\lambda)q_1q_2\) for some \(\lambda \in F^\times\), hence \(q\) is decomposable if \(\dim V_1, \dim V_2 > 1\), i.e., if \(\text{deg } A_1, \text{deg } A_2 > 1\).

If \(\sigma_1\) and \(\sigma_2\) are symplectic, then they are adjoint to some skew-symmetric bilinear forms \(b_1\) on \(V_1\) and \(b_2\) on \(V_2\). By [11, Corollary 6.10], the polar form \(bq\) of \(q\) satisfies \(bq \cong \lambda b_1 \otimes b_2\) for some \(\lambda \in F^\times\), hence \(q\) is hyperbolic. Since hyperbolic forms of dimension at least 4 are decomposable, the proof is complete.

We now use results from Section 2 to derive from (3) the decomposability of \(q\) over special fields.

**Proposition 3.3.** Suppose the adjoint involution \(\text{ad}_q\) decomposes as in (3) with \(\text{deg } A_1, \text{deg } A_2 > 1\). Then the quadratic form \(q\) is decomposable if the field \(F\) is linked.

**Proof.** Whenever \(A_1\) and \(A_2\) are split, the decomposability of \(q\) follows from Corollary 3.2. We may therefore restrict our discussion to the case where \(A_1\) and \(A_2\) are not split. The Brauer class of \(A_1\) and \(A_2\) then has order 2, hence \(\text{deg } A_1\) and \(\text{deg } A_2\) are even, and therefore \(\dim q \equiv 0 \mod 4\). Moreover, by [11, Lemma 10.25] or [5, (7.3)], \(\text{disc } q = 1\). The decomposability of \(q\) follows from Proposition 2.14.

The decomposability of \(q\) can also be derived from certain types of decompositions of \(\text{ad}_q\):

**Proposition 3.4.** Suppose the adjoint involution \(\text{ad}_q\) decomposes as in (3) with \(\text{deg } A_1, \text{deg } A_2 > 1\). Then the quadratic form \(q\) is decomposable in each of the following cases:

(a) \(A_1\) and \(A_2\) are Brauer-equivalent to a quaternion \(F\)-algebra \(Q\) and \(\sigma_1, \sigma_2\) are symplectic;

(b) \(\text{deg } A_1 = 2\) and \(\sigma_1, \sigma_2\) are orthogonal;

(c) \(\text{deg } A_1 = 4\) and \(\sigma_1, \sigma_2\) are symplectic;

(d) \(\text{deg } A_1 = 4, \sigma_1, \sigma_2\) are orthogonal and \(\text{disc } \sigma_1 = 1\);

(e) \(\text{deg } A_1 = 8, \sigma_1, \sigma_2\) are symplectic and \(A_1\) is not a division algebra;

(f) \(\text{deg } A_1 = 8, \sigma_1, \sigma_2\) are orthogonal, \(\text{disc } \sigma_1 = 1\) and one of the factors of the Clifford algebra \(C(A_1, \sigma_1)\) is split.
Moreover, in case (a) the form $q$ is divisible by the norm form $n_Q$ of $Q$, and in case (b) it is divisible by the form $(1, -\text{disc} \sigma_1)$.

**Proof.** In case (a), we may assume $Q$ is a division algebra, since the proposition follows from Corollary 3.2 if $A_1$ and $A_2$ are split. Let $\gamma$ be the conjugation involution on $Q$. Since $\sigma_1$ and $\sigma_2$ are symplectic, it follows from [11, Proposition, p. 202] or [1, Proposition 3.4] that there exist quadratic forms $q_1', q_2'$ on $F$-vector spaces $V_1', V_2'$ such that

$$(A_1, \sigma_1) \cong (Q, \gamma) \otimes_F (\text{End}_F V_1', \text{ad}_{q_1'}), \quad (A_2, \sigma_2) \cong (Q, \gamma) \otimes_F (\text{End}_F V_2', \text{ad}_{q_2'}).$$

On the other hand, by [5, (11.1)], we have

$$(Q, \gamma) \otimes_F (Q, \gamma) \cong (\text{End}_F Q, \text{ad}_{n_Q}).$$

Therefore, (3) implies

$$(\text{End}_F V, \text{ad}_q) \cong (\text{End}_F Q, \text{ad}_{n_Q}) \otimes_F (\text{End}_F V_1', \text{ad}_{q_1'}) \otimes_F (\text{End}_F V_2', \text{ad}_{q_2'}).$$

It follows by [11, Corollary 6.10] that $q \cong (\lambda)n_Q q'_1 q'_2$ for some $\lambda \in F^\times$, hence $q$ is decomposable if $\dim V_1'$ or $\dim V_2'$ is at least 2. If $\dim V_1' = \dim V_2' = 1$, then $q$ is a multiple of $n_Q$ hence it is decomposable by Proposition 2.8.

In case (b), the quaternion algebra with involution $(A_1, \sigma_1)$ becomes hyperbolic over $F(\sqrt{\text{disc} \sigma_1})$, hence $q$ also becomes hyperbolic over this extension. By Lemma 2.3, there exists a quadratic form $q'$ such that

$q \cong (1, -\text{disc} \sigma_1) \otimes q'$.

Since $\text{disc} q = 1$ by [11, Lemma 10.25] or [5, (7.3)], the dimension of $q'$ is even. Therefore, the Witt index $i_W(q)$ is even. Letting $i_W(q) = 2m$, we have

$q = q_a \perp 2m(1, -1) \cong (1, -\text{disc} \sigma_1) \otimes (q' \perp m(1, -1)).$

Cases (c)–(f) reduce to (a) or (b) since in each case there is a decomposition of the form

$$(A_1, \sigma_1) \cong (Q, \sigma) \otimes_F (A', \sigma')$$

for some quaternion $F$-algebra $Q$: see [11, Proposition 10.21] for case (c), [11, Proposition 10.26] for case (d), [2, Theorem 7] for case (e) and [5, (42.11)] for case (f). \qed

**3.2. Negative results on Shapiro’s question**

For any central simple algebra with orthogonal or symplectic involution $(A, \sigma)$ over a field $F$, the involution trace form $T_\sigma : A \rightarrow F$ is defined by

$T_\sigma = \text{Trd}_A(\sigma(x)x),$
where Trd_A is the reduced trace. By [5, (11.1)], the map \( \sigma_* : A \otimes_F A \to \text{End}_F A \) defined by \( \sigma_*(a \otimes b)(x) = ax\sigma(b) \) for \( a, b, x \in A \) carries the involution \( \sigma \otimes \sigma \) on \( A \otimes_F A \) to the adjoint involution of \( T_\sigma \), so

\[
(\text{End}_F A, \text{ad}_{T_\sigma}) \simeq (A, \sigma) \otimes (A, \sigma).
\]

Therefore, \( \text{ad}_{T_\sigma} \) is decomposable if \( \text{deg} A > 1 \). However, we show in this subsection that the quadratic form \( T_\sigma \) is indecomposable for certain orthogonal involutions \( \sigma \) on central simple algebras \( A \) of degree \( \text{deg} A \equiv 2 \mod 4 \), providing a negative solution to Shapiro’s question.

**Remark.** If the base field \( F \) is linked, Proposition 3.3 shows that the form \( T_\sigma \) is decomposable when \( \text{deg} A > 1 \), since its adjoint involution is decomposable.

We start our construction of examples where \( T_\sigma \) is indecomposable with a lemma from elementary number theory.

**Lemma 3.5.** Suppose \( m, k, l, r, s \geq 1 \) are integers such that

\[
m^2 = kl, \quad (m - 1)^2 = rs, \quad (k \geq r \geq 1, \quad l \geq s \geq 1).
\]

If \( m \equiv s \equiv 1 \mod 2 \), then \( l = 1 \).

**Proof.** The hypotheses show that \( l \) is odd. Let \( k - r = a \) and \( l - s = 2b \). By (5), we have \( k > a \geq 0 \) and \( l > 2b \geq 0 \), hence, by the first equation in (4),

\[
m^2 > 2ab \geq 0.
\]

By definition of \( a \) and \( b \),

\[
2ab = (k - r)(l - s) = (k - r)l + k(l - s) + rs - kl,
\]

hence, by (4),

\[
2ab = al + 2bk + (m - 1)^2 - m^2.
\]

It follows that

\[
al + 2bk = 2m - 1 + 2ab,
\]

hence, using the first equation in (4),

\[
(al - 2bk)^2 = (al + 2bk)^2 - 8abkl = (2m - 1 + 2ab)^2 - 8abm^2.
\]
This equation shows that

\[(2m - 1 + 2ab)^2 \geq 8abm^2,\]

hence

\[(2ab - 1)(2ab - 1 + 4m - 4m^2) \geq 0.\]  \hspace{1cm} (7)

By (6) we have

\[2ab - 1 + 4m - 4m^2 < 4m - 1 - 3m^2,\]

and the right side is easily seen to be nonpositive for all \(m \geq 1\). Therefore, (7) yields \(2ab \leq 1\), hence \(a = 0\) or \(b = 0\) since \(a\) and \(b\) are nonnegative integers. If \(a = 0\), then (4) yields

\[m^2 = kl \quad \text{and} \quad (m - 1)^2 = ks.\]

Since \(m^2\) and \((m - 1)^2\) are relatively prime, it follows that \(k = 1\), hence \(s = (m - 1)^2 \equiv 0 \mod 2\), a contradiction. Therefore, \(b = 0\) and (4) implies \(l = 1\) since \(m^2\) and \((m - 1)^2\) are relatively prime. \(\Box\)

**Theorem 3.6.** Let \(Q\) be a quaternion algebra over a field \(F\) and let \(m\) be an odd integer. If an orthogonal involution \(\sigma\) on the matrix algebra \(M_m(Q)\) satisfies the following conditions:

(a) the biquaternion algebra \(Q \otimes F (\text{disc } \sigma, -1)F\) is division,

(b) there is an ordering \(P\) on \(F\) such that \(\text{sgn}_P T_\sigma = 4(m - 1)^2,\)

then \(T_\sigma\) is an indecomposable form.

**Proof.** Suppose \(T_\sigma = q_1q_2\) for some quadratic forms \(q_1, q_2\). We use condition (a) to show that \(\dim q_1\) and \(\dim q_2\) cannot be both even, and condition (b) to show that the decomposition is trivial if one of \(\dim q_1\), \(\dim q_2\) is odd.

Suppose first \(\dim q_1 \equiv \dim q_2 \equiv 0 \mod 2\). By Lemma 2.1, the Clifford invariant \(c(q_1q_2)\) has index at most 2. On the other hand, the Hasse invariant of \(T_\sigma\) was computed by Lewis in [6] and Quéguiner in [9]. From their computation, and from the relations between the Hasse and the Clifford invariant (see [10, p. 81]), it follows that \(c(T_\sigma)\) is represented by the biquaternion algebra \(Q \otimes F (\text{disc } \sigma, -1)F\), which has index 4 if condition (a) holds. Therefore, this case leads to a contradiction.

Suppose next that \(\dim q_2\) is odd. Since \(\dim T_\sigma = 4m^2\), the dimension of \(q_1\) is a multiple of 4. Let \(\dim q_1 = 4k\) and \(\dim q_2 = l\), so

\[m^2 = kl, \quad k, l, m \geq 1 \quad \text{and} \quad m \equiv 1 \mod 2.\]
Comparing signatures at $P$, we also have

$$\text{sgn}_P T_\sigma = 4(m - 1)^2 = \text{sgn}_P q_1 \text{sgn}_P q_2,$$

with $\text{sgn}_P q_1 \equiv \dim q_1 \equiv 0 \mod 2$ and $\text{sgn}_P q_2 \equiv \dim q_2 \equiv 1 \mod 2$. Therefore, $\text{sgn}_P q_1$ is a multiple of 4. Letting $|\text{sgn}_P q_1| = 4r$ and $|\text{sgn}_P q_2| = s$, we have

$$(m - 1)^2 = rs, \quad r, s \geq 1 \quad \text{and} \quad s \equiv 1 \mod 2.$$ 

Lemma 3.5 shows that $l = 1$, so the decomposition $T_\sigma = q_1 q_2$ is trivial. \hfill \square

Let $-$ be the conjugation involution on a quaternion division $F$-algebra $Q$. For $x \in M_m(Q)$, we define $\bar{x}$ by letting $-$ act on each entry of $x$, so the map $x \mapsto \bar{x}'$ is a symplectic involution on $M_m(Q)$. To obtain orthogonal involutions, consider diagonal matrices of pure quaternions

$$\Delta = \text{diag}(\alpha_1, \ldots, \alpha_m)$$

with $\alpha_i = -\alpha_i \neq 0$ for $i = 1, \ldots, m$, and define $\sigma_\Delta : M_m(Q) \to M_m(Q)$ by

$$\sigma_\Delta(x) = \Delta^{-1} \cdot \bar{x}' \cdot \Delta \quad \text{for} \ x \in M_m(Q). \quad (8)$$

Since $\Delta^t = -\Delta$, the map $\sigma_\Delta$ is an orthogonal involution, see [5, (2.7)]. Moreover, since every orthogonal involution on the endomorphism algebra of a finite-dimensional $Q$-vector space is adjoint to some skew-hermitian form with respect to $-$ (see [5, (4.2)]), every orthogonal involution on $M_m(Q)$ is conjugate to an involution $\sigma_\Delta$ for some $\Delta$ as above.

**Proposition 3.7.** The discriminant of $\sigma_\Delta$ and the signature of $T_{\sigma_\Delta}$ at an ordering $P$ of $F$ are determined as follows:

(a) $\text{disc} \sigma_\Delta = \alpha_1^2 \cdots \alpha_m^2 F^2 F^2 \in F^2 / F^2$.
(b) Let $F_P$ be a real closure of $F$ at $P$. If $Q$ is not split by $F_P$, then

$$\text{sgn}_P T_{\sigma_\Delta} = 0.$$ 

If $Q$ is split by $F_P$, choose a pure quaternion $\beta \in Q^\times$ such that $\beta^2 < p 0$, let $m_1$ be the number of indices $i$ such that $\alpha_i^2 < p 0$ and $\text{Trd}_Q(\alpha_i \beta) > p 0$, and let $m_2$ be the number of indices $i$ such that $\alpha_i^2 < p 0$ and $\text{Trd}_Q(\alpha_i \beta) < p 0$; then

$$\text{sgn}_P T_{\sigma_\Delta} = 4(m_1 - m_2)^2.$$
**Proof.** The arguments\(^1\) in the proof of [5, (7.3)(2)] show that

\[
\text{disc } \sigma \Delta = (-1)^m \text{Nrd}_{M_m(Q)}(\Delta) F^{\times 2} = (-1)^m \text{Nrd}_Q(\alpha_1) \cdots \text{Nrd}_Q(\alpha_m) F^{\times 2}.
\]

Part (a) follows, since \(\alpha_i^2 = -\text{Nrd}_Q(\alpha_i)\).

If \(Q\) is not split by \(F_P\), then \(\text{sgn}_P T_{\sigma \Delta} = 0\) by [7, Theorem 1] (or [5, (11.11)]). For the rest of the proof, suppose \(Q\) is split by \(F_P\). For \(x = (x_{ij})_{1 \leq i, j \leq m} \in M_m(Q)\) we have \(\sigma \Delta(x) = (x'_{ij})_{1 \leq i, j \leq m}\) where

\[
x'_{ij} = \alpha_i^{-1} x_{ji} \alpha_j.
\]

Therefore,

\[
T_{\sigma \Delta}(x) = \sum_{i, j=1}^m \text{Trd}_Q(\alpha_i^{-1} x_{ji} \alpha_j x_{ji}).
\]

Letting \(T_{ij} : Q \rightarrow F\) denote the quadratic form

\[
T_{ij}(x) = \text{Trd}_Q(\alpha_i^{-1} x \alpha_j x),
\]

we thus have

\[
T_{\sigma \Delta} = \bigoplus_{1 \leq i, j \leq m} T_{ij},
\]

hence

\[
\text{sgn}_P T_{\sigma \Delta} = \sum_{i, j=1}^m \text{sgn}_P T_{ij}.
\]

If \(\alpha_i\) and \(\alpha_j\) anticommute, then 1 and \(\alpha_i\) span a totally isotropic subspace of \(Q\) for \(T_{ij}\), so \(T_{ij}\) is hyperbolic and \(\text{sgn}_P T_{ij} = 0\). Note that in this case we have either \(\alpha_i^2 > 0\) or \(\alpha_j^2 > 0\) since \(Q\) splits over \(F_P\). If \(\alpha_i\) and \(\alpha_j\) do not anticommute, pick a nonzero quaternion \(\alpha'\) which anticommutes with \(\alpha_i\). Computation shows that \((1, \alpha_i, \alpha_j, \alpha_i \alpha_j, \alpha_i \alpha'\) is an orthogonal basis of \(Q\) for \(T_{ij}\), and that \(T_{ij}\) has the following diagonalization in this basis:

\[
T_{ij} = \langle T(\alpha_i^{-1} \alpha_j) \rangle \langle 1, -\alpha_i^2, -\alpha_j^2, \alpha_i^2 \alpha_j^2 \rangle.
\]

---

\(^1\) There is a misprint in the statement of [5, Proposition (7.3)(2)]: the formula given there yields \(\text{det}(\text{Int}(u) \circ \sigma)\) instead of \(\text{disc}(\text{Int}(u) \circ \sigma)\).
Note that $\alpha^2 > p$ 0 if $\alpha_i^2 < p$ 0 since $Q$ splits over $F_P$. It follows that $\text{sgn}_p T_{ij} = 0$ unless $\alpha_i^2 < p$ 0 and $\alpha_j^2 < p$ 0; in this case, $\text{sgn}_p T_{ij} = \pm 4$. Therefore, letting

$$t_{ij} = \begin{cases} 
0 & \text{if } \alpha_i^2 > p \ 0 \text{ or } \alpha_j^2 > p \ 0, \\
1 & \text{if } \alpha_i^2 < p \ 0, \alpha_j^2 < p \ 0 \text{ and } \text{Trd}_Q(\alpha_i^{-1} \alpha_j) > p \ 0, \\
-1 & \text{if } \alpha_i^2 < p \ 0, \alpha_j^2 < p \ 0 \text{ and } \text{Trd}_Q(\alpha_i^{-1} \alpha_j) < p \ 0,
\end{cases}$$

we have

$$\text{sgn}_p T_{\sigma \Delta} = 4 \sum_{i,j=1}^{m} t_{ij}.$$ 

Now, let $\beta \in Q^\times$ be a pure quaternion such that $\beta^2 < p$ 0. If a pure quaternion $\alpha \in Q$ satisfies $\text{Trd}_Q(\alpha \beta) = 0$, then $\beta$ anticommutes with $\alpha$, hence $\alpha^2 > p$ 0 since $Q$ splits over $F_P$. For $i = 1, \ldots, m$, let

$$s_i = \begin{cases} 
0 & \text{if } \alpha_i^2 > p \ 0, \\
1 & \text{if } \alpha_i^2 < p \ 0 \text{ and } \text{Trd}_Q(\alpha_i \beta) > p \ 0, \\
-1 & \text{if } \alpha_i^2 < p \ 0 \text{ and } \text{Trd}_Q(\alpha_i \beta) < p \ 0,
\end{cases}$$

so that

$$\sum_{i=1}^{m} s_i = m_1 - m_2.$$ 

To complete the proof, we have to show

$$\sum_{i,j=1}^{m} t_{ij} = \left( \sum_{i=1}^{m} s_i \right)^2.$$ 

It clearly suffices to prove $t_{ij} = s_i s_j$ for $i, j = 1, \ldots, m$, which amounts to showing that if $\alpha_i^2 < p$ 0 and $\alpha_j^2 < p$ 0, then $\text{Trd}_Q(\alpha_i^{-1} \alpha_j) > p$ 0 if and only if $\text{Trd}_Q(\alpha_i \beta)$ and $\text{Trd}_Q(\alpha_j \beta)$ have the same sign. Since $\alpha_i^2 = -\text{Nrd}_Q(\alpha_i)$ and $\text{Trd}_Q(\alpha_i \alpha_j) = \alpha_i^2 \text{Trd}_Q(\alpha_i^{-1} \alpha_j)$, it is equivalent to show that if $\text{Nrd}_Q(\alpha_i) > p$ 0 and $\text{Nrd}_Q(\alpha_j) > p$ 0, then

$$\text{Trd}_Q(\alpha_i \beta) \text{Trd}_Q(\alpha_j \beta) \text{Trd}_Q(\alpha_i \alpha_j) < p \ 0.$$ 

To check this statement, we may extend scalars to a real closure of $F$ at $P$. We may therefore substitute for $Q$ a matrix algebra over a real-closed field. The following lemma completes the proof:
Lemma 3.8. Let $K$ be an ordered field and let $\alpha, \alpha', \beta \in M_2(K)$ be matrices with trace zero. If $\det \alpha, \det \alpha', \det \beta > 0$, then $\text{tr}(\alpha \alpha') \text{tr}(\alpha \beta) \text{tr}(\alpha' \beta) < 0$. □

We are indebted to the referee for the following simple proof.

Proof. Let $V \subset M_2(K)$ be the vector space of matrices of trace zero. The restriction of the determinant is a quadratic form $q$ on $V$ isometric to $\langle 1, -1, -1 \rangle$, and its polar form is $-\frac{1}{2} \text{tr}(\xi \eta)$. Consider the matrix

$$G = \begin{pmatrix}
\det \alpha & -\frac{1}{2} \text{tr}(\alpha \alpha') & -\frac{1}{2} \text{tr}(\alpha \beta) \\
-\frac{1}{2} \text{tr}(\alpha \alpha') & \det \alpha' & -\frac{1}{2} \text{tr}(\alpha' \beta) \\
-\frac{1}{2} \text{tr}(\alpha \beta) & -\frac{1}{2} \text{tr}(\alpha' \beta) & \det \beta
\end{pmatrix}.$$

If $\alpha, \alpha', \beta$ are not linearly independent, $\det G = 0$. Otherwise, $G$ is the Gram matrix of $q$ with respect to the basis $\alpha, \alpha', \beta$, hence $\det G > 0$. Therefore,

$$\det \alpha \det \alpha' \det \beta - \frac{1}{4} \text{tr}(\alpha \alpha') \text{tr}(\alpha \beta) \text{tr}(\alpha' \beta) \geq \frac{1}{4} \text{tr}(\alpha \alpha')^2 \det \beta + \frac{1}{4} \text{tr}(\alpha \beta)^2 \det \alpha'.
\tag{9}$$

Similarly, since $V$ does not contain any positive-definite 2-dimensional subspace,

$$\det \begin{pmatrix}
\det \alpha & -\frac{1}{2} \text{tr}(\alpha \alpha') \\
-\frac{1}{2} \text{tr}(\alpha \alpha') & \det \alpha'
\end{pmatrix} \leq 0,$$

hence

$$\det \alpha \det \alpha' \det \beta \leq \frac{1}{4} \text{tr}(\alpha \alpha')^2 \det \beta.
\tag{10}$$

From (9) and (10), it follows that

$$-\frac{1}{4} \text{tr}(\alpha \alpha') \text{tr}(\alpha \beta) \text{tr}(\alpha' \beta) \geq \frac{1}{4} \text{tr}(\alpha \beta)^2 \det \alpha' + \frac{1}{4} \text{tr}(\alpha' \beta)^2 \det \alpha \geq 0.$$

Moreover, (10) shows that $\text{tr}(\alpha \alpha') \neq 0$. By symmetry, we also have $\text{tr}(\alpha \beta) \neq 0$ and $\text{tr}(\alpha' \beta) \neq 0$, and the proof is complete. □

Remark. It follows from [7, Theorem 1] that the signature of the involution trace form $T_{\sigma}$ is a square for any orthogonal or symplectic involution $\sigma$ on a central simple algebra $A$. In the terminology of [7] (see also [5, (11.10)]), Proposition 3.7(b) states that if $Q$ is split by $F_{p}$, then $\text{sgn}_p \sigma \Delta = |m_1 - m_2|$.
We proceed to give examples satisfying the hypotheses of Theorem 3.6, for any odd integer \( m \geq 3 \).

Let \( F_0 \) be an arbitrary ordered field and let \( F = F_0(x, y) \), the field of rational fractions in two indeterminates over \( F_0 \). The following classical construction (see [8, p. 75]) extends the ordering on \( F_0 \) to an ordering \( P \) on \( F \) such that \( x <_P 0 \) and \( 0 <_P y <_P 1 \): consider the \((x, y)\)-adic valuation on \( F \) defined on \( F_0[x, y] \setminus \{0\} \) by

\[
v\left( \sum_{i,j} a_{ij} x^i y^j \right) = \min \{ (i, j) \mid a_{ij} \neq 0 \},
\]

where the minimum is for the lexicographic order on \( \mathbb{Z}^2 \). If \( f \in F^\times \) satisfies \( v(f) = (m, n) \), then \( v((-x)^{-m} y^{-n} \bar{f}) = 0 \) and we may consider the residue \((-x)^{-m} y^{-n} \bar{f} \in F_0 \). Set

\[
f >_P 0 \text{ if and only if } (-x)^{-m} y^{-n} \bar{f} > 0 \text{ in } F_0.
\]

(11)

Then \(-x >_P 0, y >_P 0 \) and \( 1 - y >_P 0 \), so \( x <_P 0 \) and \( 0 <_P y <_P 1 \).

Consider the quaternion \( F \)-algebra \( Q = (x, y)_F \). For any odd integer \( m \geq 3 \), consider the following pure quaternions in \( Q \):

\[
\alpha_1 = i + k, \quad \alpha_2 = j, \quad \alpha_3 = \cdots = \alpha_m = i.
\]

and let \( \sigma_\Delta \) be the involution on \( M_m(Q) \) defined as in (8). Since \( y >_P 0 \), the algebra \( Q \) is split by any real closure of \( F \) at \( P \). To compute \( \text{sgn}_P T_{\sigma_\Delta} \), we use Proposition 3.7 with \( \beta = i \). Since

\[
\alpha_1^2 = x(1 - y) <_P 0, \quad \alpha_2^2 = y >_P 0 \quad \text{and} \quad \text{Trd}_Q(i(i + k)) = \text{Trd}_Q(i^2) = 2x <_P 0,
\]

we have, in the notation of Proposition 3.7, \( m_1 = 0 \) and \( m_2 = m - 1 \), so

\[
\text{sgn}_P T_{\sigma_\Delta} = 4(m - 1)^2.
\]

By Proposition 3.7, we also have

\[
\text{disc}_\Delta = \alpha_1^2 \cdots \alpha_m^2 F^\times_2 \cdot (x - xy) y x^{m-2} F^\times_2 = (1 - y) y F^\times_2.
\]

To see that the hypotheses of Theorem 3.6 hold, it remains to prove that the tensor product

\[
(x, y)_F \otimes_F (y(1 - y), -1)_F
\]

is a division algebra, or, equivalently by a theorem of Albert (see [5, (16.5)]), that an associated Albert form such as

\[
q = \langle x, y, -xy, y(y - 1), y(y - 1), 1 \rangle
\]
is anisotropic. Since
\[ q = \langle 1, y, y(y - 1), y(y - 1) \rangle \perp \langle x \rangle \langle 1, -y \rangle, \]
a degree argument (see for instance [4, Lemma 1.4(i)]) shows that \( q \) is isotropic over \( F \) if and only if one of the forms \( \langle 1, y, y(y - 1), y(y - 1) \rangle, \langle 1, -y \rangle \) is isotropic over \( F_0(y) \). The latter is clearly anisotropic since \( y \notin F_0(y)^x_2 \). Using the \( (y - 1) \)-adic valuation on \( F_0(y) \), one may construct as above (see (11)) an ordering \( P_0 \) on \( F_0(y) \) such that \( y > P_0 1 \). Then
\[ \text{sgn}_{P_0} \langle 1, y, y(y - 1), y(y - 1) \rangle = 4, \]
so \( \langle 1, y, y(y - 1), y(y - 1) \rangle \) is anisotropic over \( F_0(y) \).

Thus, Theorem 3.6 shows that the quadratic form \( T_{\sigma \Delta} \) is indecomposable, even though its adjoint involution is decomposable, as it is isomorphic to \( \sigma_\Delta \otimes \sigma_\Delta \).

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